The $T\bar{T}$ Deformation of Quantum Field Theory

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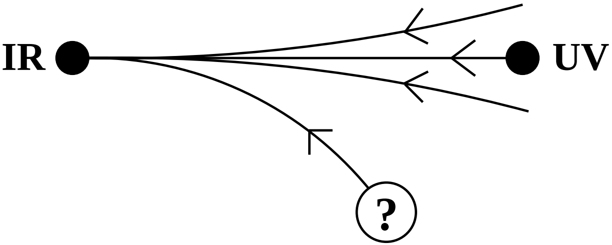
ICMP, Montreal, July 2018

Introduction

- all QFTs that we use in physics are in some sense *effective* field theories, valid over only some range of energy/length scales
- if they are (perturbatively) renormalizable, this range of scales may be very large and they have more predictive power, but eventually new physics should enter
- if they are non-renormalizable (the action involves operators with dimension $> d$) they may still be useful up to some energy scale $\sim$ UV cut-off Λ
- even so they may still make sense at higher energies if they have a 'UV completion', *eg.* if they flow from a non-trivial RG fixed point – 'asymptotic safety'

However another possibility is that the UV limit is not a conventional UV fixed point corresponding to a local QFT, but is something else (*eg* string theory):



The $T\overline{T}$ deformation of 2d QFT is an example of a non-renormalizable deformation of a local QFT for which, however, many physical quantities make sense and are *finite* and calculable in terms of the data of the undeformed theory.

However this deformation is very special – this has been termed 'asymptotic fragility,' which could be used as a constraint on physical theories.

What is $T\overline{T}$?

- consider a sequence of 2d euclidean field theories $\mathcal{T}^{(t)}$ ($t \in \mathbb{R}$) in a domain endowed with a flat euclidean metric η_{ij}, each with a local stress-energy tensor

$$T_{ij}^{(t)}(x) \sim \delta S^{(t)}/\delta g^{ij}(x)$$

- $\mathcal{T}^{(0)}$ is a conventional local QFT (massive, or massless (CFT))
- deformation is defined formally by

$$\boxed{S^{(t+\delta t)} = S^{(t)} - \delta t \int \det T^{(t)}\, d^2x}$$

  equivalently by inserting $\int \det T^{(t)} d^2x$ into correlation functions
- $\Rightarrow$ note that this uses $T^{(t)}$, not $T^{(0)}$ $\Leftarrow$

$$\det T = \frac{1}{2}\epsilon^{ik}\epsilon^{jl}T_{ij}T_{kl} \propto T_{zz}T_{\bar{z}\bar{z}} - T_{z\bar{z}}^2$$ in complex coordinates

- for a CFT this is $T_{zz}T_{\bar{z}\bar{z}} \equiv T\overline{T}$
- since this has dimension 4, we would expect $\langle \det T \rangle \sim \Lambda^4$
- Zamolodchikov (2004) pointed out that by conservation $\partial^i T_{ij} = 0$ that

$$\frac{\partial}{\partial y_m}\epsilon^{ik}\epsilon^{jl}T_{ij}(x)T_{kl}(x+y) = \frac{\partial}{\partial x_i}\epsilon^{mk}\epsilon^{jl}T_{ij}(x)T_{kl}(x+y)$$

so that in any translationally invariant state

$$\boxed{\langle \epsilon^{ik}\epsilon^{jl}T_{ij}(x)T_{kl}(x)\rangle = \langle \epsilon^{ik}\epsilon^{jl}T_{ij}(x)T_{kl}(x+y)\rangle}$$

- finite, and calculable in terms of matrix elements of T_{ij}
- so the deformation is in some sense ‘solvable’

$T\overline{T}$ as a topological deformation

In this talk I'll describe a different approach which also works in non-translationally invariant geometries

$$e^{(\delta t/2)\int \epsilon_{ik}\epsilon_{jl}T^{ij}(x)T^{kl}(x)d^2x} = \int [dh_{ij}]e^{-(1/2\delta t)\int \epsilon^{ik}\epsilon^{jl}h_{ij}(x)h_{kl}(x)d^2x+\int h_{ij}T^{ij}d^2x}$$

- $h_{ij} = O(\delta t)$ may be viewed as an infinitesimal change in the metric $g_{ij} = \eta_{ij} + h_{ij}$
- in 2d we can always write

$$h_{ij} = a_{i,j} + a_{j,i} + \delta_{ij}\Phi$$

where a_i is an infinitesimal diffeomorphism $x_i \to x_i + a_i(x)$ and $e^{\Phi} \sim 1 + \Phi$ is the conformal factor

- however, at the saddle point $h = h^*[T]$ (which is sufficient since the integral is gaussian)

  $$T^{ij} \propto \epsilon^{ik}\epsilon^{jl}h^*_{kl}$$

  conservation $\partial_i T^{ij} = 0$ then implies that $\Box\Phi = 0$, so the metric is *flat*
- Φ can be absorbed into the diffeomorphism
- moreover we can take $a_{i,j} = a_{j,i}$

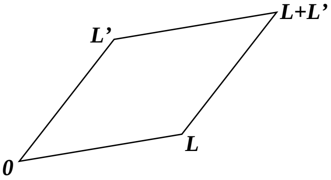
- the action is then

$$(2/\delta t)\int \epsilon^{ik}\epsilon^{jl}(\partial_i a_j)(\partial_k a_l)d^2x - 2\int(\partial_i a_j)T^{ij}d^2x$$

$$= (2/\delta t)\int \partial_i(\epsilon^{ik}\epsilon^{jl}a_j\partial_k a_l)d^2x - 2\int \partial_i(a_jT^{ij})d^2x$$

  and so is *topological*:
- for a simply connected domain only a boundary term
- for a closed manifold, only contributions from nontrivial windings of a_i
  - only $h_{ij} = 2a_{i,j}$ needs to be single valued

Torus



- torus made by identifying opposite edges of a parallelogram with vertices at $(0, L, L', L+L')$ in $\mathbb{C}$
- saddle point is translationally invariant

$$h^{*ij} = \delta t \, \epsilon^{ik}\epsilon^{jl}\langle T_{kl}(0)\rangle = \delta t \, \epsilon^{ik}\epsilon^{jl}(1/A)\left(L_k\partial_{L_l} + L'_k\partial_{L'_l}\right)\log Z^{(t)}$$

($A = L \wedge L' =$ area)

- change in $\log Z^{(t)}$ is

$$\int \langle T_{ij}(x) h^{*ij}[T]\rangle_c d^2x = (\delta t)\epsilon^{ik}\epsilon^{jl}\left(L_i\partial_{L_j} + L'_i\partial_{L'_j}\right)\langle T_{kl}(0)\rangle$$

Evolution equation for partition function

$$\frac{\partial}{\partial t} Z^{(t)}(L, L') = \epsilon^{ik}\epsilon^{jl} \left(L_k \partial_{L_l} + L'_k \partial_{L'_l} \right) (1/A) \left(L_k \partial_{L_l} + L'_k \partial_{L'_l} \right) Z^{(t)}(L, L')$$

In terms of $\mathfrak{Z}^{(t)} \equiv Z^{(t)}/A$

$$\boxed{\partial_t \mathfrak{Z} = (\partial_L \wedge \partial_{L'})\mathfrak{Z}}$$

- simple linear PDE, first order in ∂_t
- if $\log Z \sim -f_t A$,

$$\partial_t f_t = -f_t^2 \quad \Rightarrow \quad f_t = \frac{f_0}{1 + f_0 t}$$

– no new UV divergences in the vacuum energy

interpretation as a stochastic process

$$\partial_t \mathfrak{Z} = (\partial_L \wedge \partial_{L'})\mathfrak{Z}$$

is of diffusion type, where $\mathfrak{Z}^{(t)}$ is the pdf for a Brownian motion (L_t, L'_t) in moduli space with

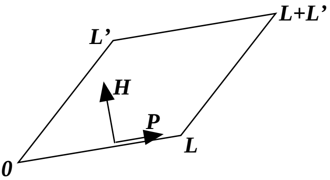
$$\mathbb{E}\left[(L_{t_1} - L_{t_2}) \wedge (L'_{t_1} - L'_{t_2})\right] = |t_1 - t_2|$$

In particular the mean area $\mathbb{E}[L_t \wedge L'_t] \sim t$ as $t \to +\infty$, with, however, absorbing boundary conditions on $L \wedge L' = 0$.

The relation between the two approaches is

$$\boxed{\mathfrak{Z}^{(t)}(L_0, L'_0) = \mathbb{E}\left[\mathfrak{Z}^{(0)}(L_t, L'_t)\right]}$$

Finite-size spectrum



$$
\begin{aligned}
Z(L,L') &= \mathrm{Tr}\, e^{-(R\,\mathrm{Im}\,\tau)\hat{H}(R)+i(R\,\mathrm{Re}\,\tau)\hat{P}(R)} \\
&= \sum_n e^{-(R\,\mathrm{Im}\,\tau)E_n^{(t)}(R)+i(R\,\mathrm{Re}\,\tau)P_n(R)}
\end{aligned}
$$

where $R = |L|$, $\tau = L'/L$ and $\hat{H}$, $\hat{P}$ are the energy and momentum operators for the theory defined on a circle of circumference R.

PDE for $Z^{(t)}$ then leads after some algebra to [Zamolodchikov 2004]

$$
\boxed{\partial_t E_n^{(t)}(R) = -E_n^{(t)}(R)\partial_R E_n^{(t)}(R) - P_n^2/R}
$$

For $P_n = 0$ this is the inviscid Burgers equation.

If $\mathcal{T}^{(0)}$ is a CFT,

$$E_n^{(0)}(R) = 2\pi\tilde{\Delta}_n/R \quad \text{where} \quad \tilde{\Delta}_n = \Delta_n - c/12$$

Solution is then (with $P_n = 0$)

$$E_n^{(t)}(R) = \frac{R}{2t}\left(1 - \sqrt{1 - \frac{8\pi\tilde{\Delta}_n\, t}{R^2}}\right)$$

- energies with $\tilde{\Delta}_n > 0$ become singular at some finite $t > 0$
- energies with $\tilde{\Delta}_n < 0$ become singular at some finite $t < 0$

Thermodynamics

- identifying $R \equiv \beta = 1/kT$,
  $E_0^{(t)}(\beta) = \beta f_t(\beta)$, where $f_t =$ free energy per unit length
- for fixed $t < 0$ there is a transition at finite $T \sim 1/(-ct)^{1/2}$
- this is of Hagedorn type where the density of states grows exponentially
- if $\mathcal{T}^{(0)}$ is a free boson, then $E_n^{(t)}(R, P)$ is the spectrum of the Nambu-Goto string [Caselle *et al.* 2013] which is known to have a Hagedorn transition as a world sheet theory
- on the other hand for $t > 0$ the free energy is analytic but the energy density $\mathcal{E}$ is finite as $T \to \infty$, suggesting another branch with negative temperature

S-matrix

- if $\mathcal{T}^{(0)}$ is a massive QFT, the single particle mass spectrum M is not affected by the deformation
- the 2-particle energies for $MR \gg 1$ have the expected form

$$E = 2\sqrt{M^2 + P^2}$$

  where however P is quantized according to $PR + \delta(P) \in 2\pi\mathbb{Z}$, where $\delta^{(t)}(P)$ is the scattering phase shift
- consistency with the evolution equation then requires

$$\delta^{(t)} = \delta^{(0)} - t\,M^2 \sinh\theta \quad \text{where} \quad P = M \sinh(\theta/2)$$

- this is equivalent to a CDD factor in the 2-particle S-matrix [Smirnov-Zamolodchikov 2017]

$$\boxed{S^{(t)}(\theta) = e^{-itM^2 \sinh\theta}\, S^{(0)}(\theta)}$$

- if $\mathcal{T}^{(0)}$ is integrable, so is $\mathcal{T}^{(t)}$, and applying Thermodynamic Bethe Ansatz to the deformed S-matrix yields the expected form for $E_n^{(t)}(R)$ [Cavaglià *et al.* 2016]
- in fact this dressing of the S-matrix works for non-integrable theories as well: [Dubovsky *et al.* 2012, 2013]

$$S^{(t)}(\{p\}) = e^{-i(t/8)\sum_{a<b}\epsilon_{ij}p_a^i p_b^j}\,S^{(0)}(\{p\})$$

- it corresponds to the dressing of the original theory by Jackiw-Teitelboim [1985, 1983] (topological) gravity:

$$S^{(t)} = S(g_{ij},\text{matter}) + \int\sqrt{-g}(\phi R - \Lambda)d^2x$$

where $\Lambda \sim t^{-1}$

- the torus partition function of this theory has been computed and shown to satisfy the PDE of the $T\overline{T}$ deformed theory [Dubovsky *et al.* 2017, 2018]

Simply connected domain

- boundary action is

$$(1/8\delta t)\oint \epsilon^{jl}(a_j\partial_k a_l)ds^k - 2\oint \epsilon_{ik}a_jT^{ij}ds^k - \lambda\oint a_k ds^k$$

  $\lambda =$ lagrange multiplier enforcing $\oint a_k ds^k = 0$
- fermion on boundary: coupling to T simplifies with conformal boundary condition $T_{\perp\|} = 0$
- gaussian integration then gives

$$\delta\log Z \propto \delta t \oint\oint_{|s-s'|<\ell/2} G(s-s')\langle T_{\perp\perp}(s)T_{\perp\perp}(s')\rangle_c\,\epsilon_{kl}\,ds^k ds^l$$

  where $\ell =$ perimeter and $G(s-s') = \frac{1}{2}\text{sgn}(s-s') - (s-s')/\ell$

- for a disk $|x| < R$, we may decompose into modes $a_\perp(\theta) = \sum_n a_n e^{in\theta}$
- the $n = 0$ mode gives the evolution equation

$$\partial_t Z = (1/4\pi)(\partial_R - 1/R)\partial_R Z$$

where $R = \text{perimeter}/2\pi$

- corresponds to the stochastic (Bessel) process

$$\partial_t R = -\frac{1}{4\pi R} + \eta(t)\,, \quad \overline{\eta(t')\eta(t'')} = \frac{1}{2\pi}\delta(t' - t'')$$

- curvature driven dynamics as in 2d coarsening

Other directions

- in the holographic AdS/CFT correspondence, deforming the boundary CFT with $t > 0$ has been argued to be equivalent to going into the bulk of AdS_3 by a distance $O(\sqrt{t})$ [McGough *et al.* 2016]
- in Minkowski space CFT, adding $T\overline{T}$ to the action corresponds to soft left-right scattering: $t > 0$ (resp. < 0) corresponds to attractive (repulsive) interaction and superluminal (subliminal) propagation of light signals in a background with finite energy density [Cardy 2016]
- $T\overline{T}$ has also been argued to lead to the formation of shocks in the hydrodynamic effective theory [Bernard-Doyon 2015]

Summary

- the $T\overline{T}$ (more properly, $\det T$) deformation of a local 2d QFT gives a computable example of a nonlocal UV completion of an effective field theory
- many physical quantities (partition functions, spectrum, S-matrix) are UV finite, but local operators do not appear the make sense
- it is solvable because in a sense it is topological, and it also appears to be related to a dressing of the theory by (topological) gravity
- it would be interesting to find similar deformations in higher dimensions