Analytic limit shapes for the 5-vertex model

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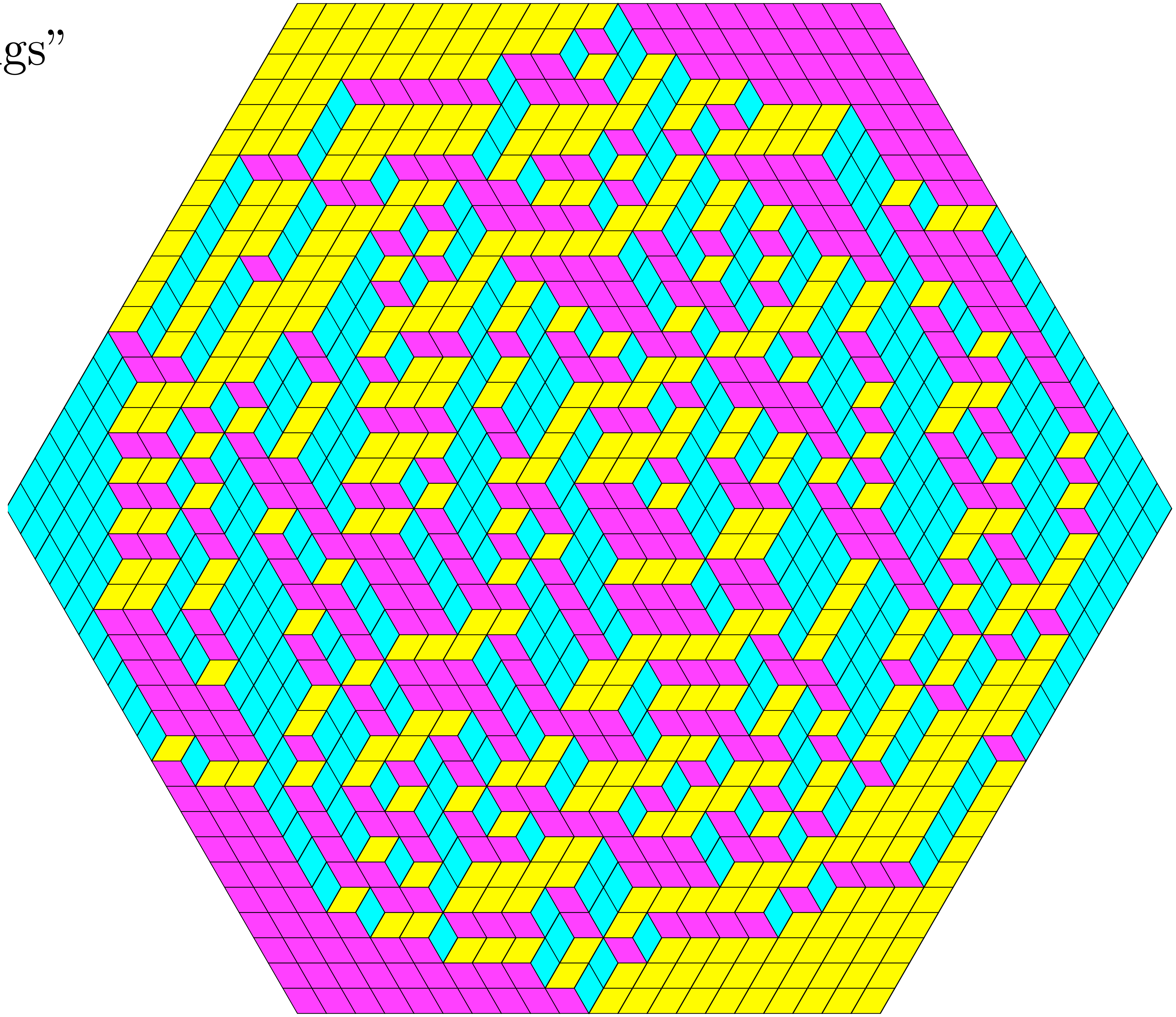
Based on joint work with

H. Cohn, J. Propp, A. Okounkov, S. Sheffield, J. de Gier, S. Watson

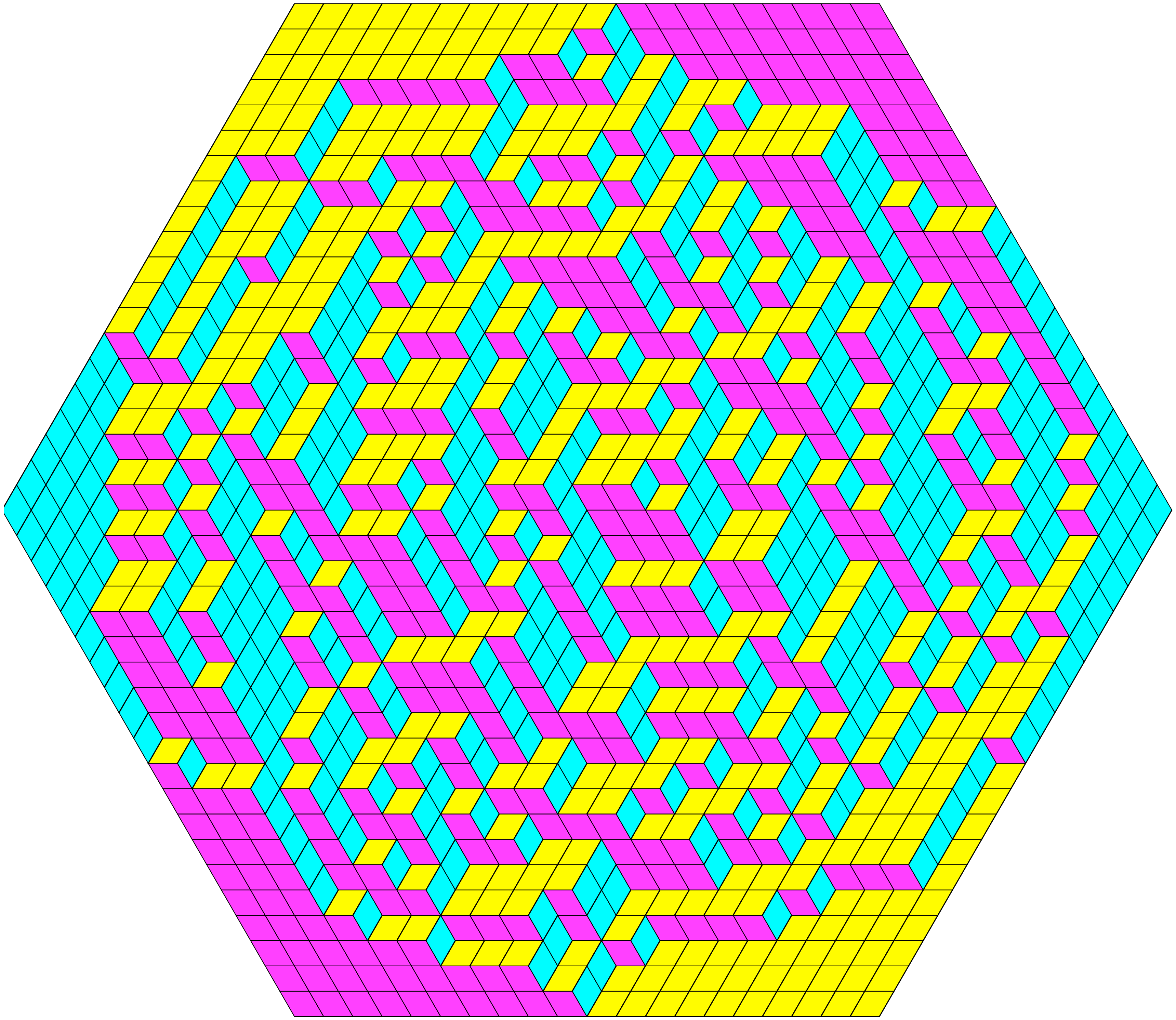
Weierstrass-Enneper parameterization of minimal surfaces

$$Re\left(\int f(z)(1-g(z)^2)\,dz, i\int f(z)(1+g(z)^2)\,dz, \int f(z)g(z)dz\right)$$

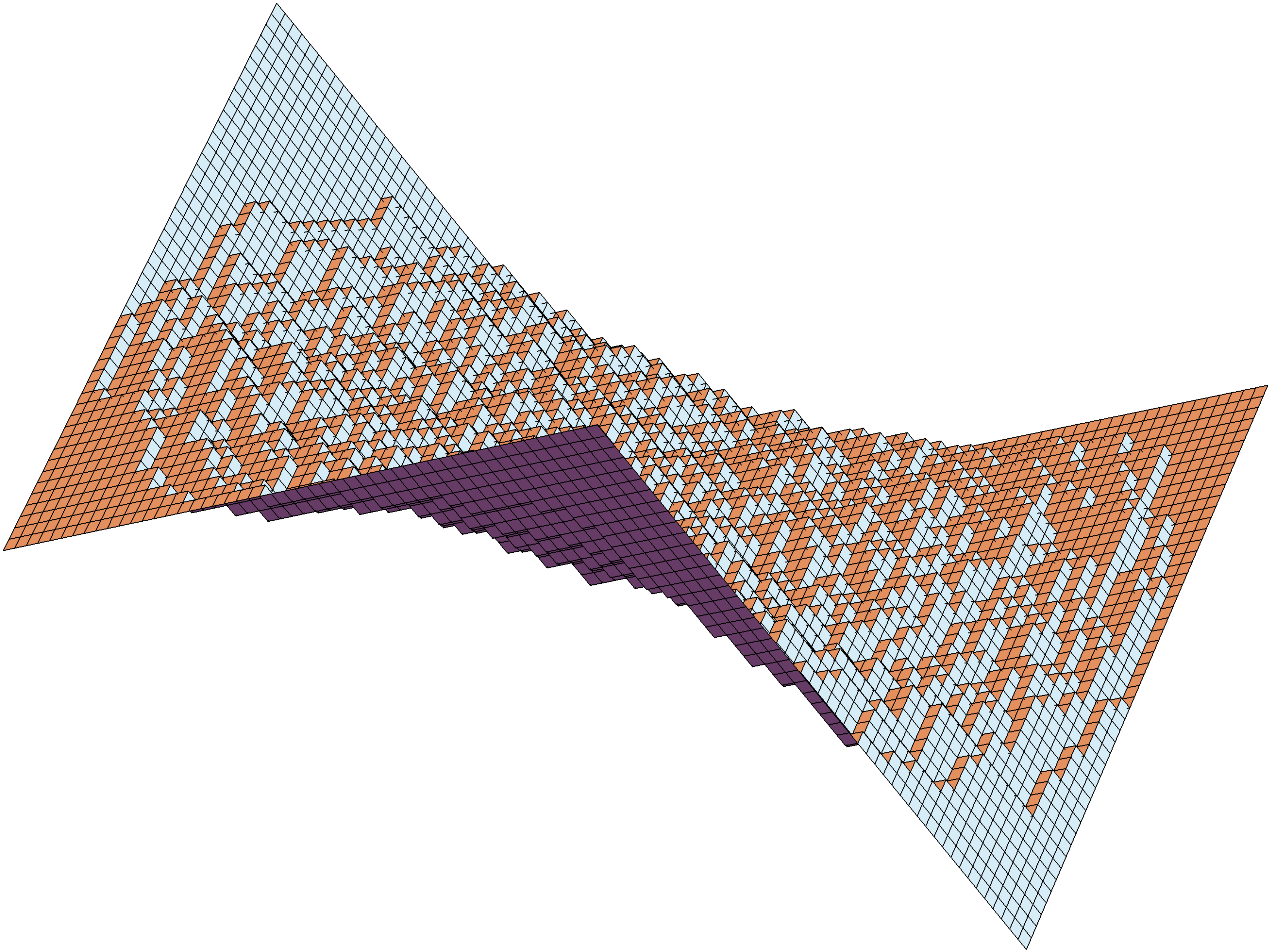
“lozenge tilings”



MacMahon (1916): $$N = \prod_{1 \le i,j,k \le n} \frac{i+j+k-1}{i+j+k-2}$$



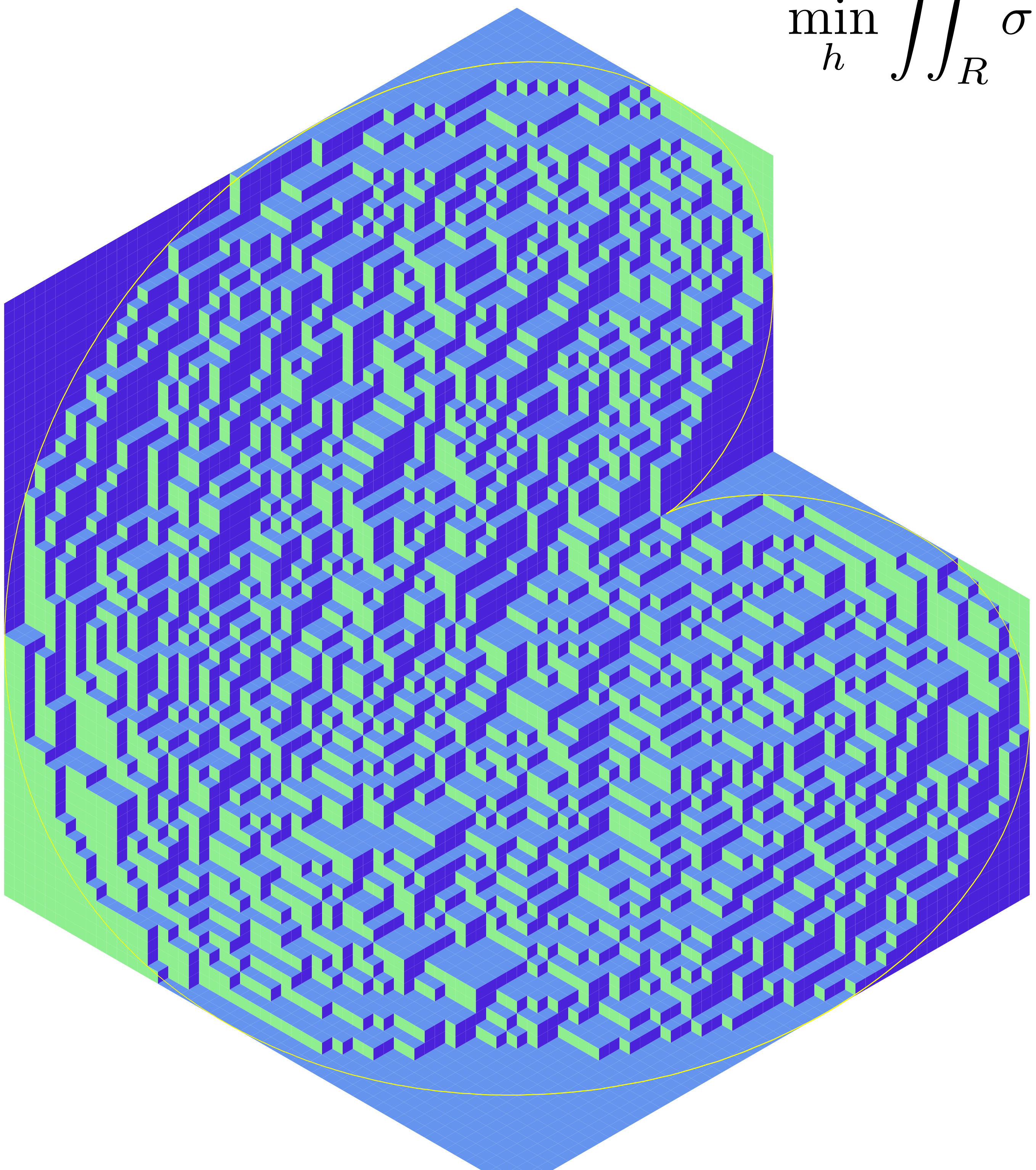
"lozenge tilings" also satisfy a variational principle



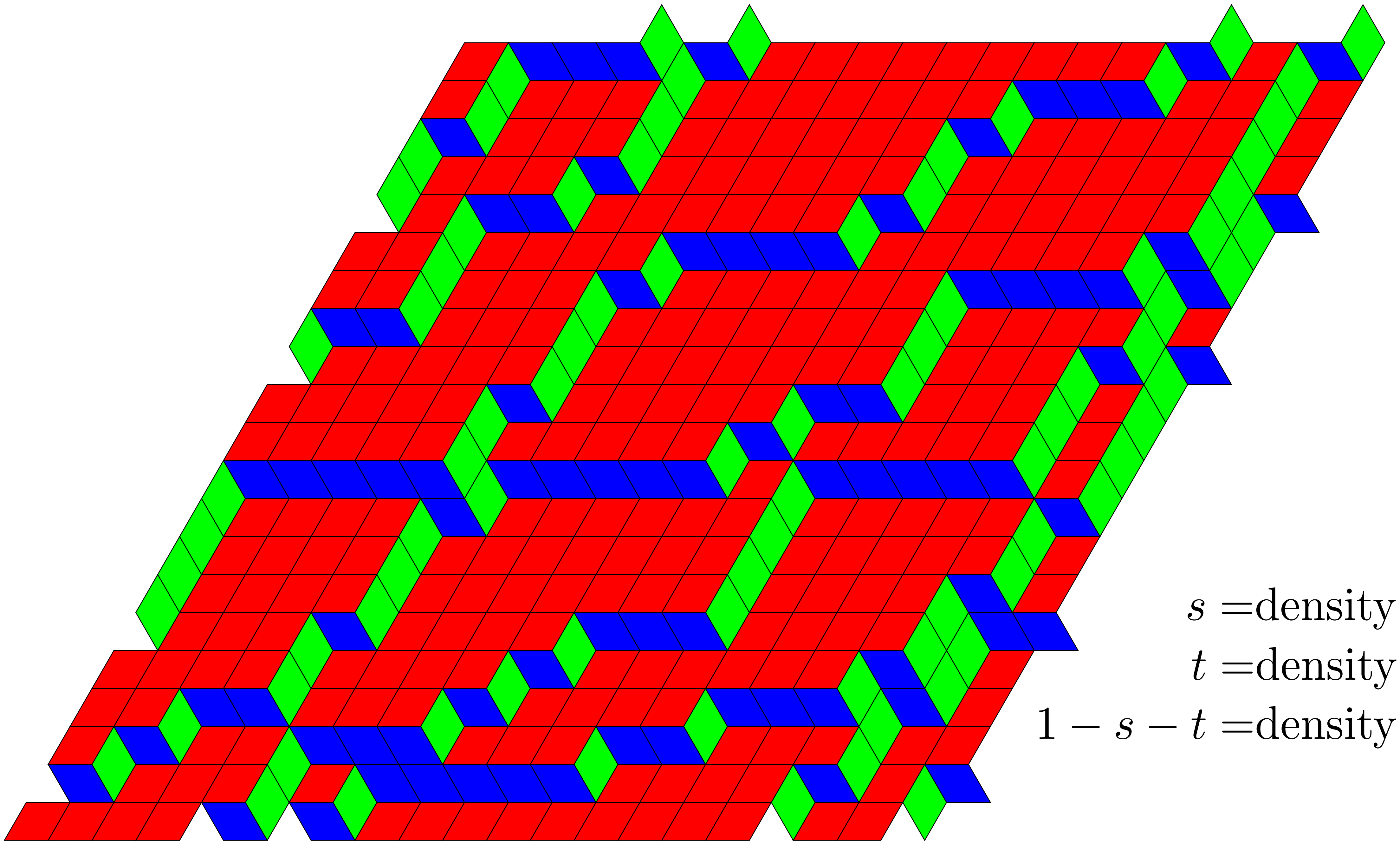
Lozenge tiling limit shape

Thm[Cohn,K,Propp (2000)] The function $h : R \to \mathbb{R}$ describing the limit shape is the unique minimizer of the surface tension integral

$$\min_h \iint_R \sigma(\nabla h)\, dx\, dy.$$



$\sigma =$ "surface tension"



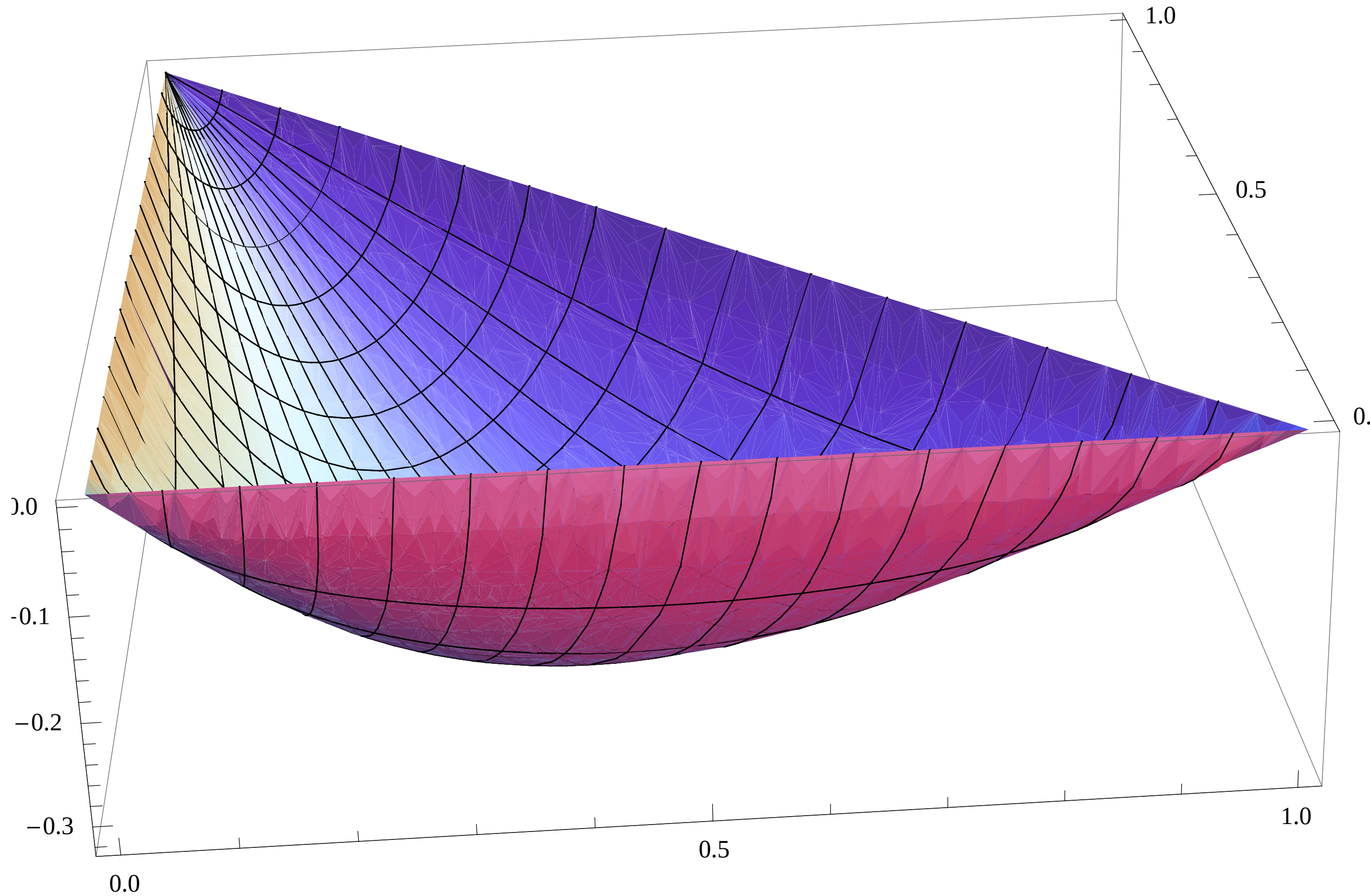
s =density of green

t =density of blue

$1 - s - t$ =density of red

for each slope (s, t) there is an associated growth rate (entropy) $-\sigma(s, t)$:

$$(\text{Number of tilings}) = e^{-\text{Area}\cdot\sigma(s,t)(1+o(1))}$$

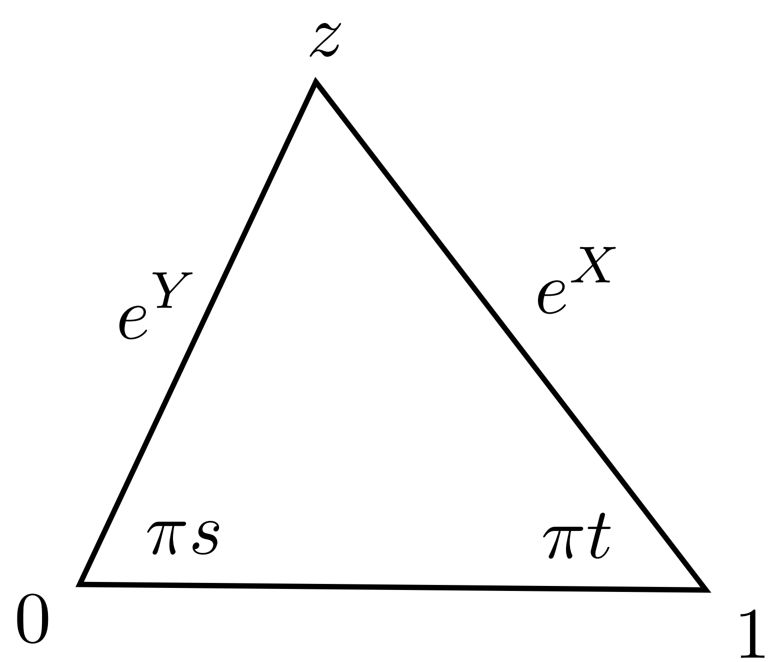


The surface tension $\sigma(s,t)$

$\sigma(s,t)$ is the *Legendre dual* of the free energy $F(X,Y)$, where tile weights are $\{1, e^X, e^Y\}$.

The surface tension $\sigma(s,t)$ can be defined by the following property:

In the triangle,



we have $$\frac{dX}{dt} = \frac{dY}{ds}$$

...so there is a function σ such that

$$\frac{d}{ds}\sigma(s,t) = X$$
$$\frac{d}{dt}\sigma(s,t) = Y$$

In terms of z,

$$\sigma(s,t) = D(z),$$

the Bloch-Wigner dilogarithm:

$$D(z) = \arg(1-z)\log|z| + \mathrm{Im}(\mathrm{Li}_2(z))$$

How to solve the variational problem?

The Euler-Lagrange equation is

$$\operatorname{div}(\nabla\sigma(\nabla h)) = c$$

or, in terms of X, Y

$$X_x + Y_y = c.$$

we can (magically) combine this with the "mixed partials" equation

$$s_y = t_x$$

to get a complex equation in terms of z (with $w = 1 - z$):

$$\frac{z_x}{z} + \frac{w_y}{w} = c.$$

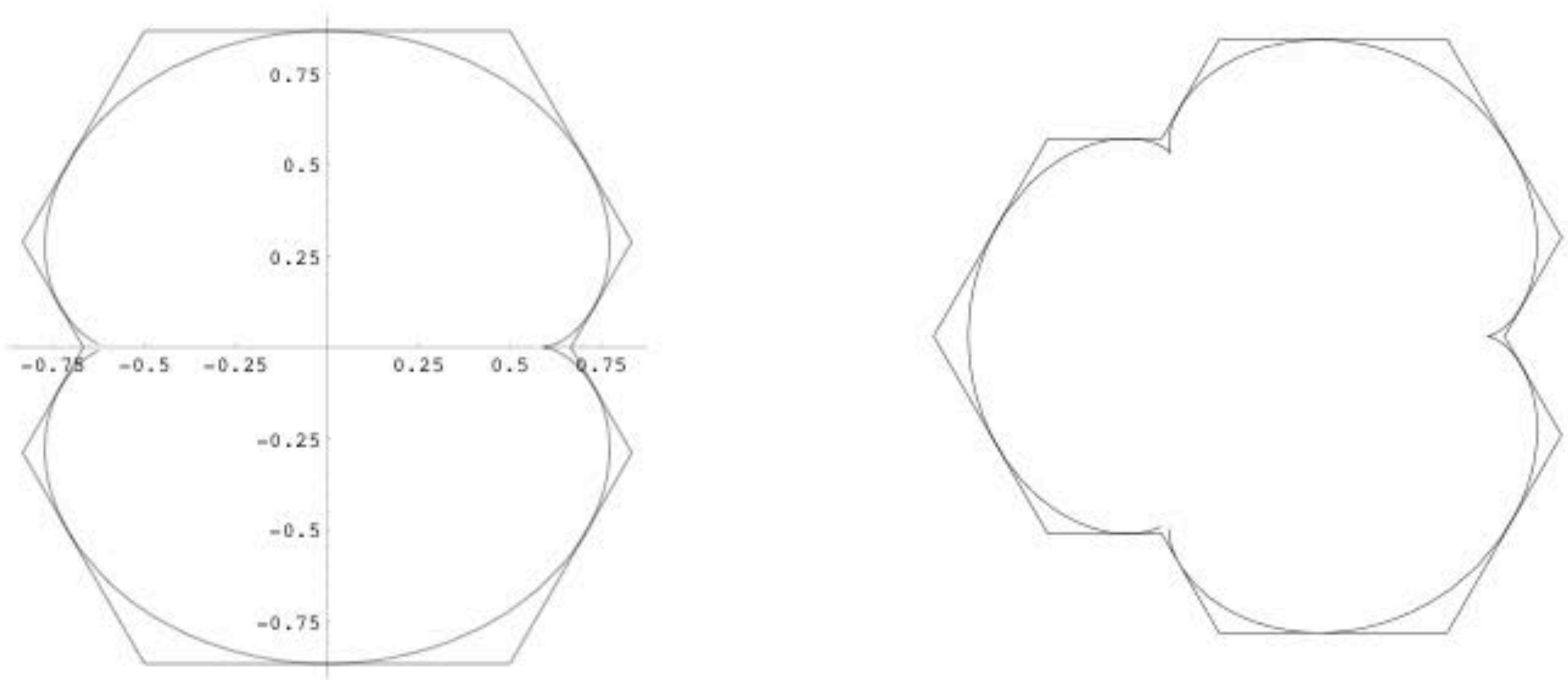
a version of the complex Burgers' equation

Thm[KO]: Solutions $z = z(x, y)$ are defined by $Q(e^{-cx}z, e^{-cy}w) = 0$ for (arbitrary) analytic Q.

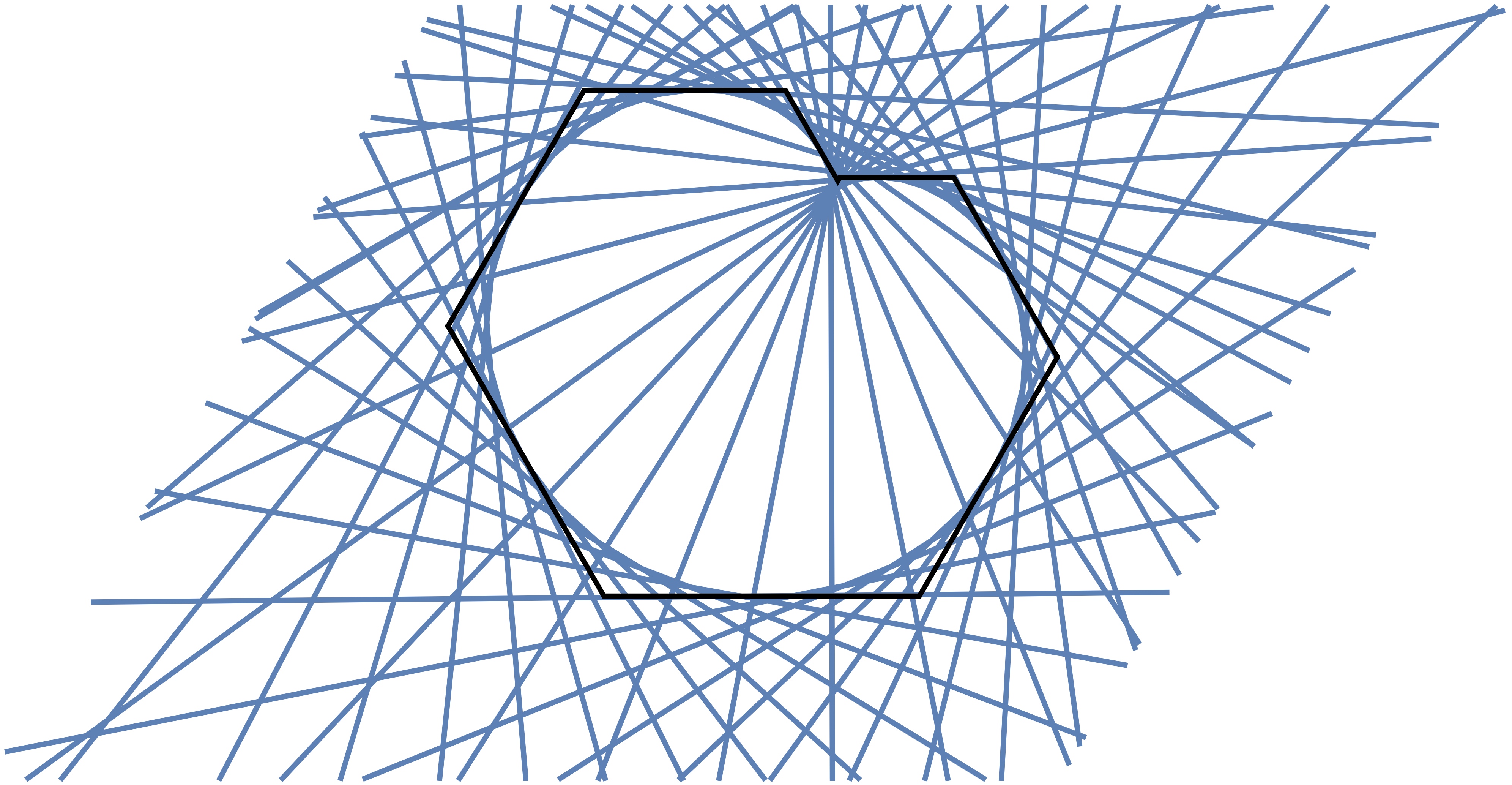
For a given wire frame Ω, how to find Q?

Theorem [KO]

When the boundary Ω is a polygonal curve with edges in directions $\hat{x}, \hat{y}, \hat{z}$, then Q is a rational curve.



The "frozen boundary" is the dual curve Q^*.



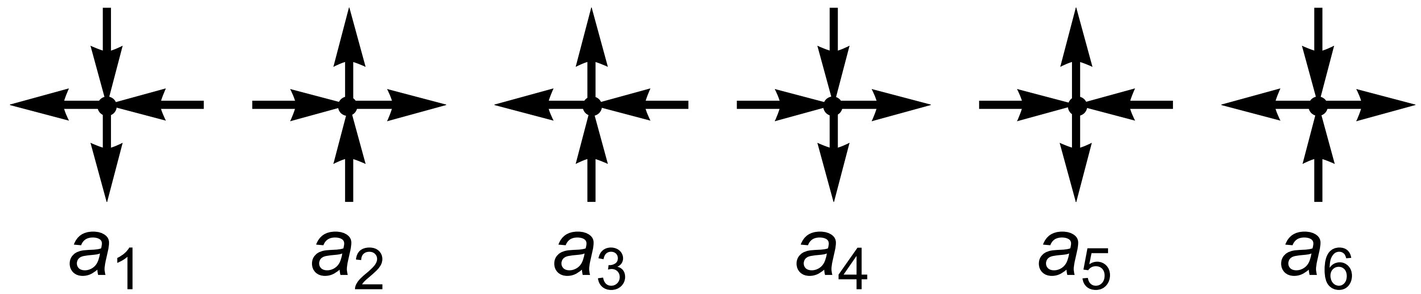
The arctic boundary is the envelope of a pencil of lines containing all boundary edges in order.

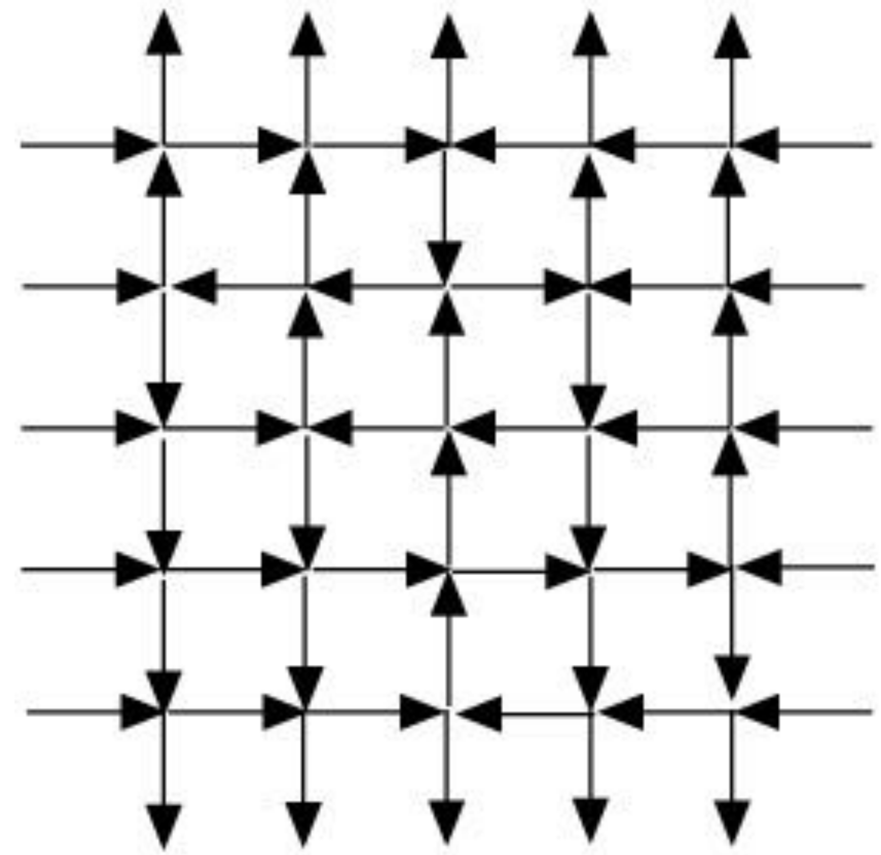
$$x\frac{(t-a_1)(t-a_2)(t-a_3)}{(t-c_1)(t-c_2)(t-c_3)}+y\frac{(t-b_1)(t-b_2)(t-b_3)}{(t-c_1)(t-c_2)(t-c_3)}=1$$

$$a_1<b_1<c_1<a_2<\cdots<c_3$$

The six-vertex model

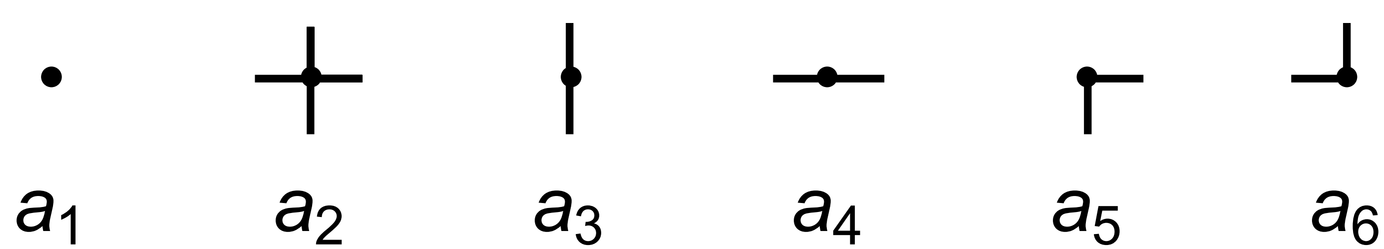
“Maps from $\mathbb{Z}^2 \to \mathbb{Z}$”





Lieb (1967): $a_1 = a_2, a_3 = a_4, a_5 = a_6$

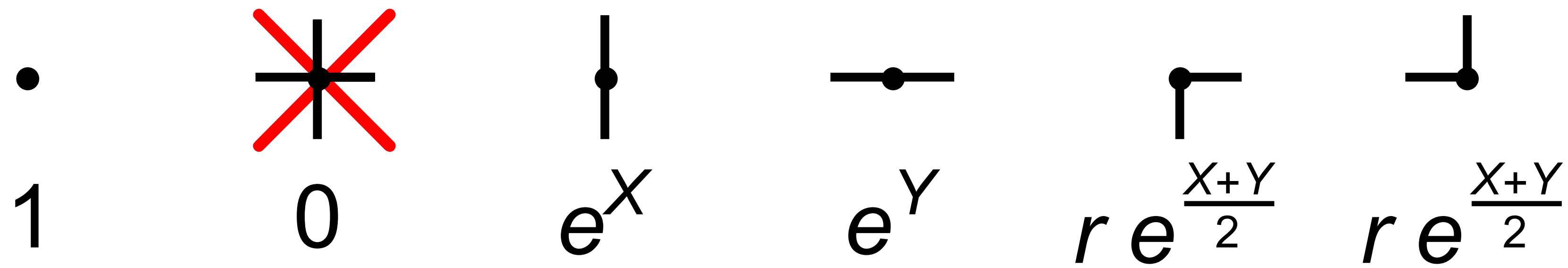
“free fermions” $a_1 a_2 + a_3 a_4 - a_5 a_6 = 0$



The five vertex model: a generalization of the lozenge tiling model
a special case of the six-vertex model $(\Delta \to \infty)$

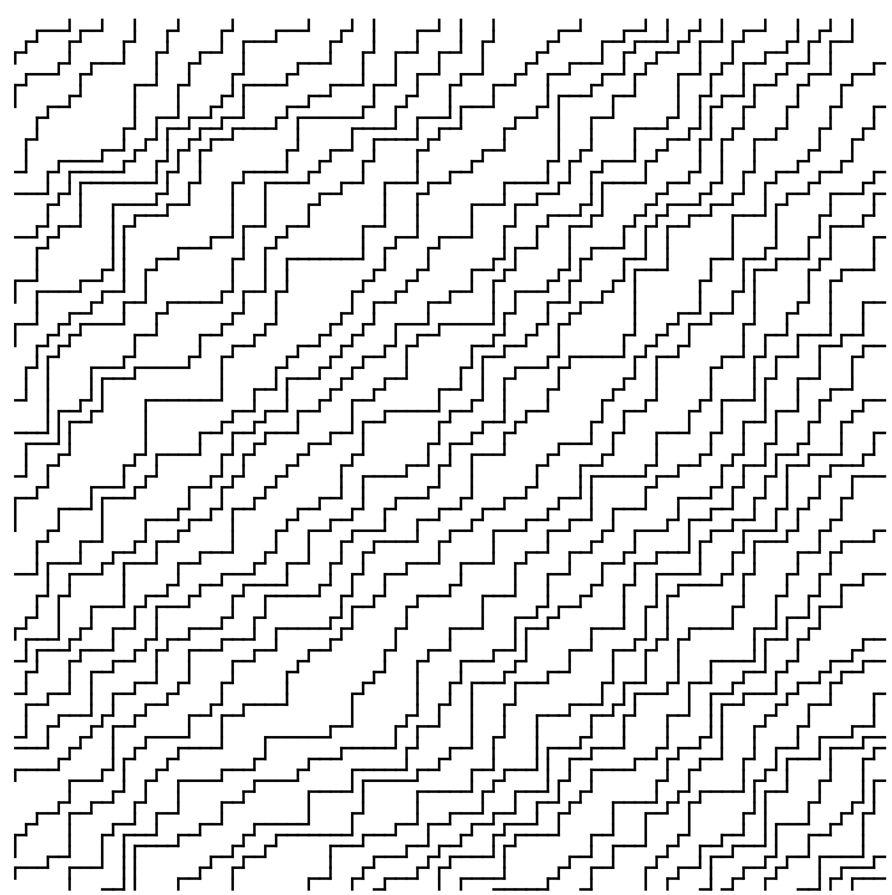
(joint with J. de Gier, S. Watson)

The five-vertex model

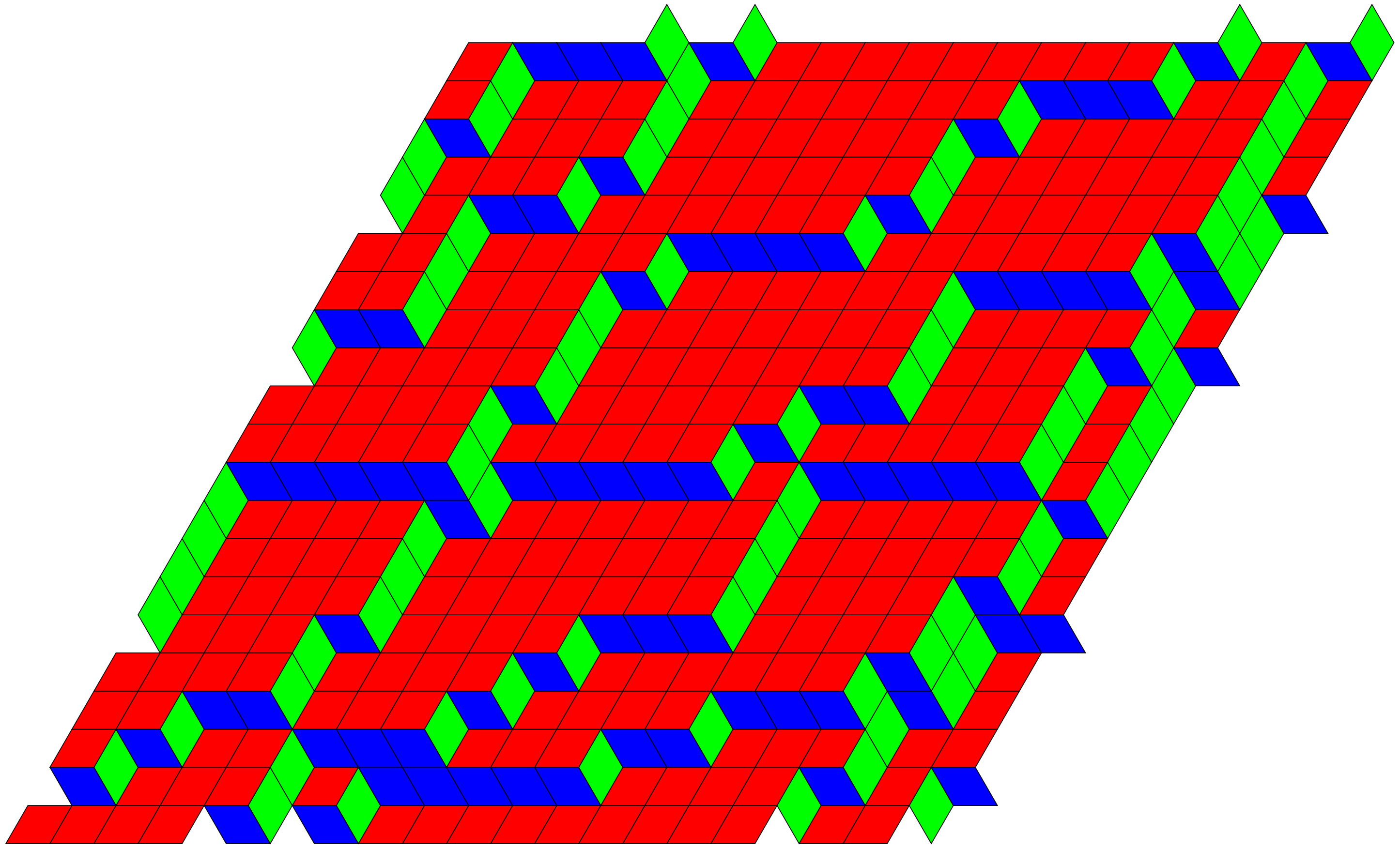


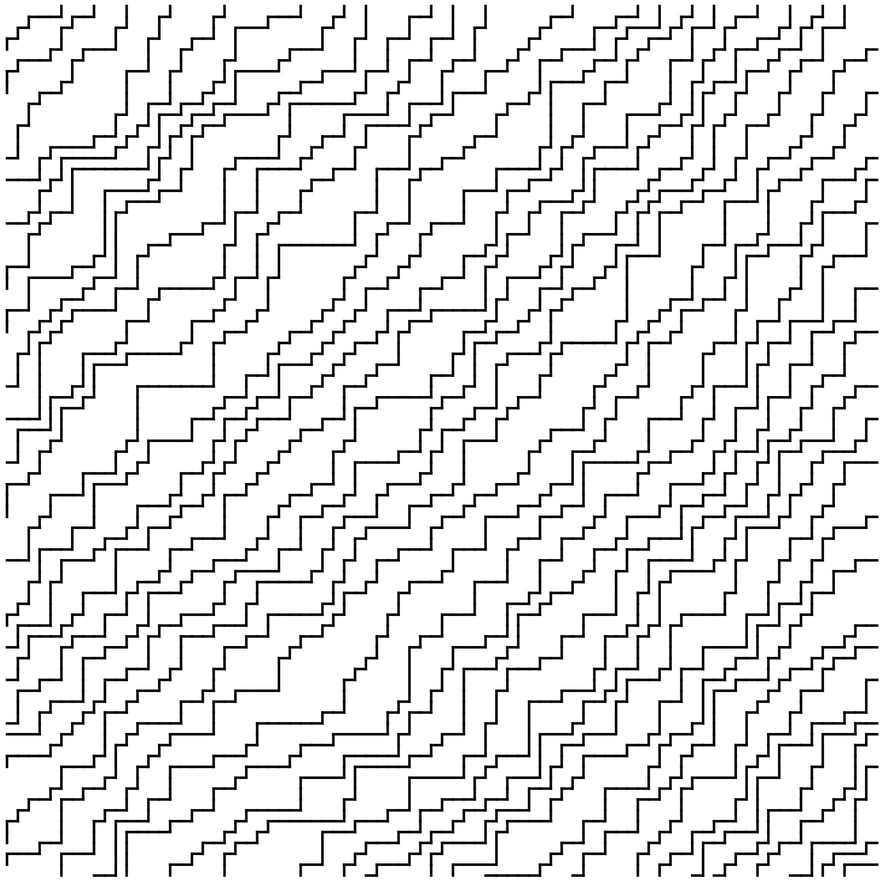
A configuration has probability $\frac{1}{Z} e^{vX+hY} r^c$ where r is the number of corners, v is the number of vertical edges, h is the number of horizontal edges.

$X = 0, Y = 0, r = 1$:



lozenge tilings and the 5-vertex model

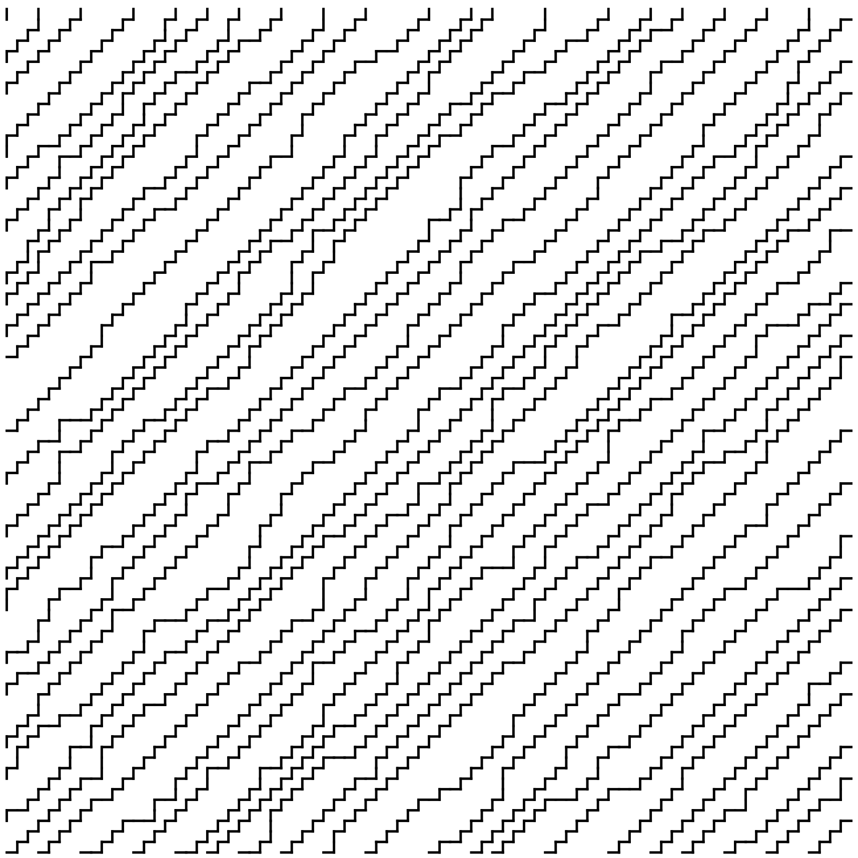




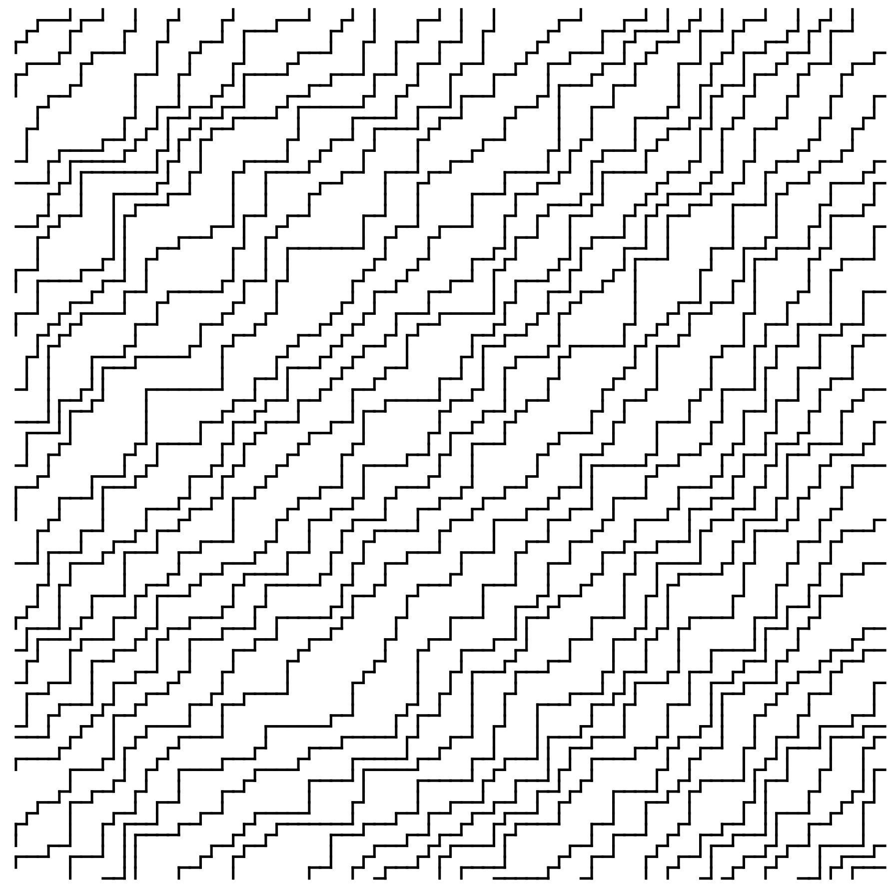
The 5 vertex model with $r = 1$ is the lozenge tiling model.

$r \neq 1$ means blue and green lozenges "interact".

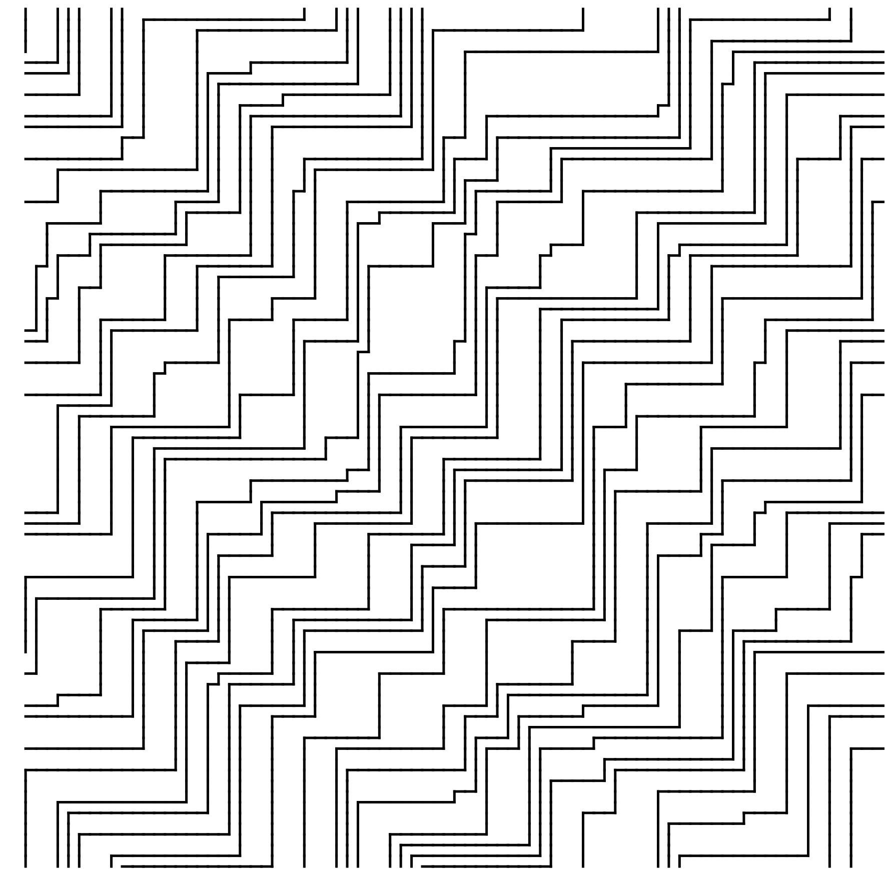
Simulations



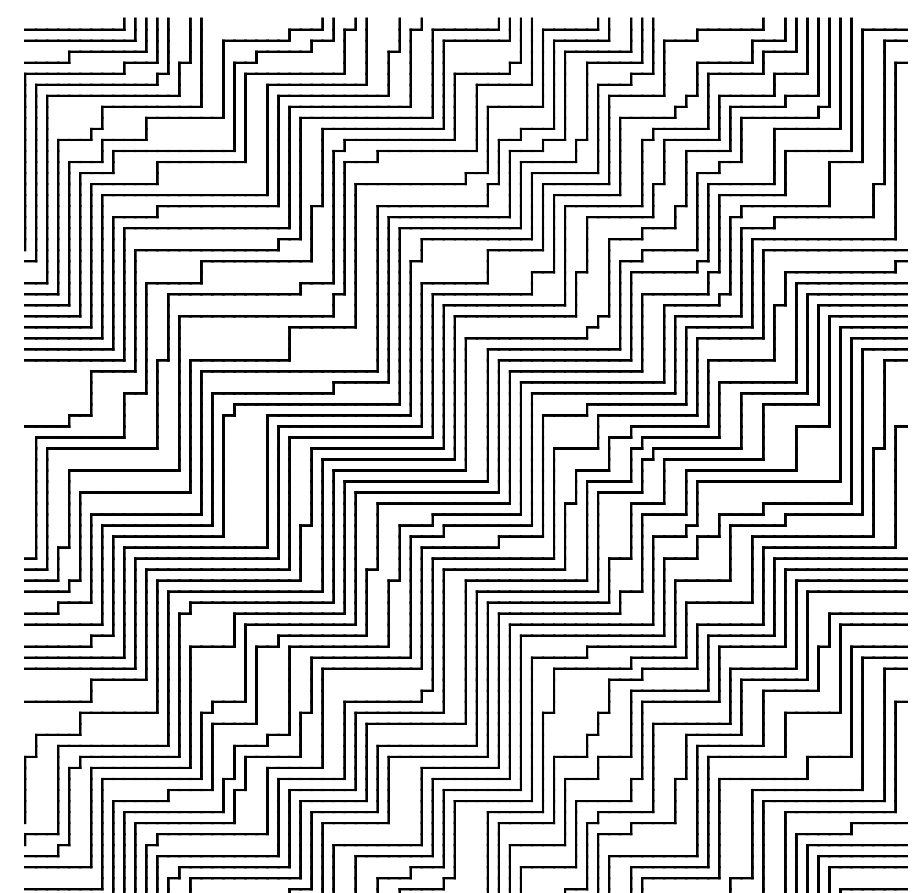
$r = 10$



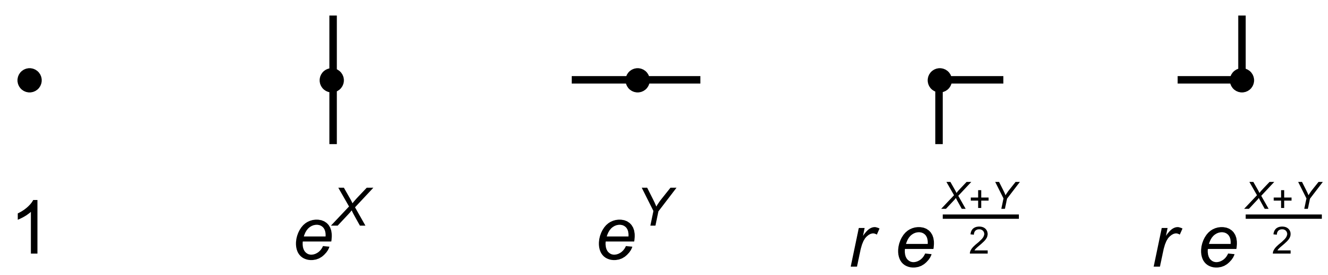
$r = 1$



$r = .1$



X, Y play the role of a "magnetic field";



For fixed r, varying (X, Y) corresponds to varying *density* (s, t) of lines

$$\begin{aligned} s &= \text{horizontal density} \\ t &= \text{vertical density} \end{aligned}$$

However the relationship between (X, Y) and (s, t) is far from trivial.

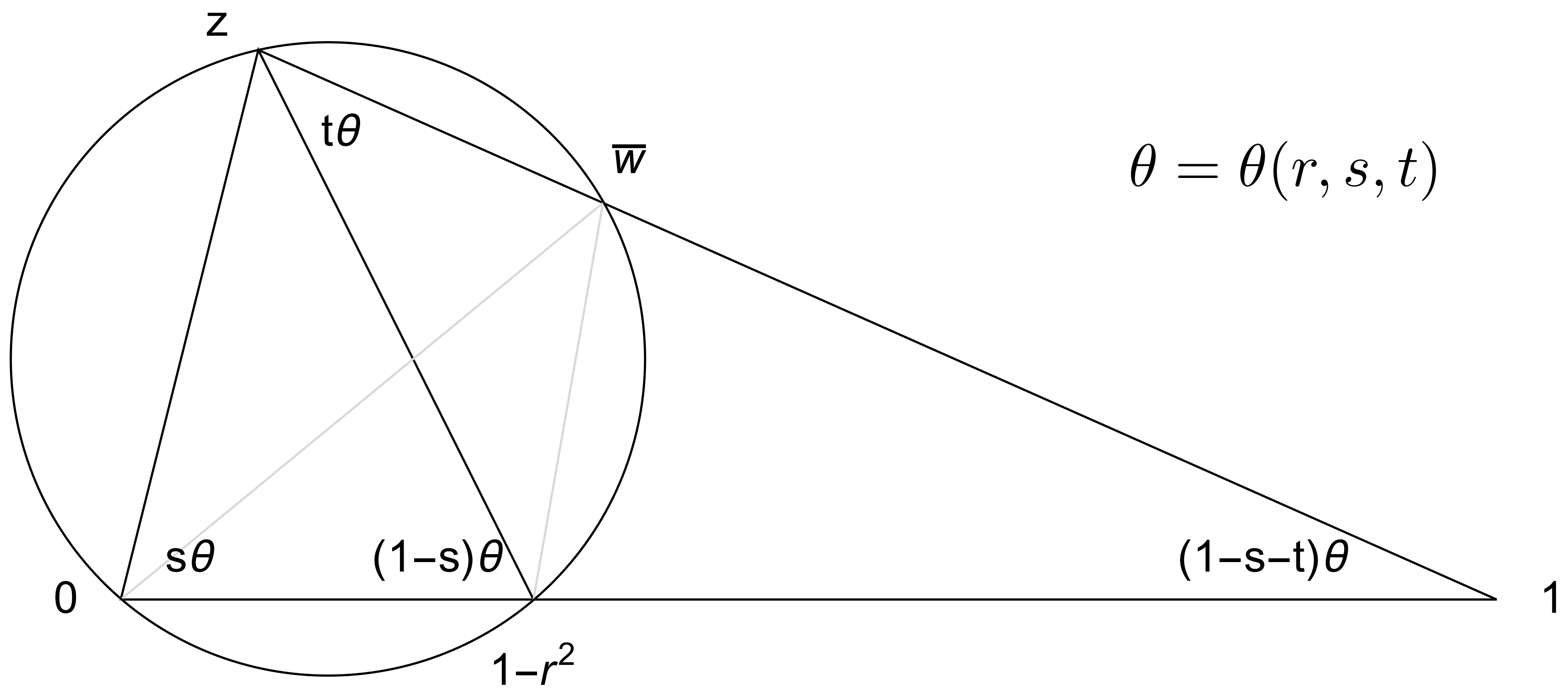
In fact knowing this relationship is equivalent to knowing the free energy:

$$\nabla F(X, Y) = (s, t).$$

Equivalently,

$$\nabla \sigma(s, t) = (X, Y).$$

$\nabla\sigma(s,t) = (X,Y)$, where X,Y are defined as follows. (Case $r<1$)



$\theta = \theta(r,s,t)$

Note: $(z-1)(w-1) = r^2$

$$X = -\log(1-r^2) - B(\frac{z}{1-r^2})$$

$$Y = -\log(1-r^2) - B(\frac{\overline{w}}{1-r^2})$$

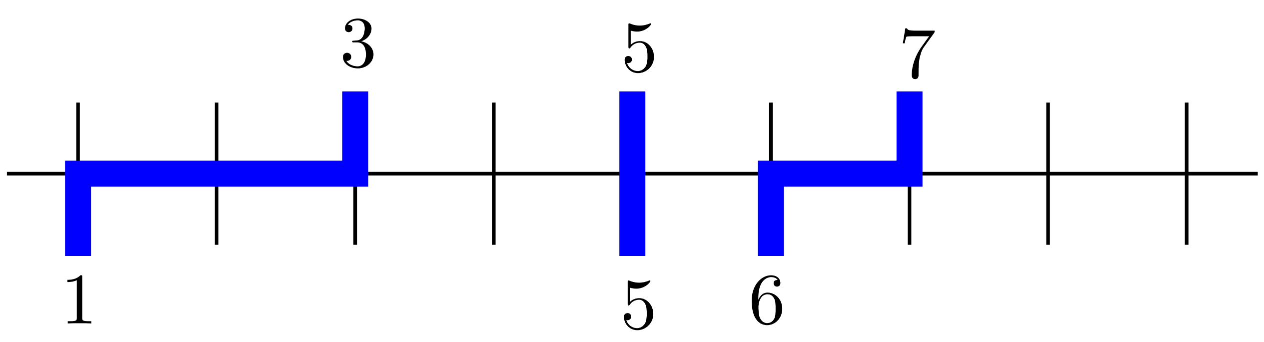
$$B(u) = \frac{1}{\pi}(\arg(u)\log|1-u| + \text{Im Li}(u))$$

(NOT the Bloch-Wigner dilog!)

General r case: how to find the free energy $F(X,Y)$?

No determinant formula...need to use Bethe Ansatz

that is, find an explicit diagonalization of the $2^N \times 2^N$ transfer matrix T



$$T(\{1,5,6\},\{3,5,7\}) = r^4$$

T has a partial diagonalization into blocks T_k:

$$T = \begin{pmatrix} T_0 & & & \\ & T_1 & & \\ & & \ddots & \\ & & & T_N \end{pmatrix}$$

T_k is the $\binom{n}{k} \times \binom{n}{k}$ transfer matrix for k particles

For T_1, eigenvectors have the form $f_\zeta(x) = \zeta^x$ where $\zeta^N = 1$.

For T_2, eigenvectors have the form $f_{\zeta_1,\zeta_2}(x_1, x_2) = A_{12}\zeta_1^{x_1}\zeta_2^{x_2} + A_{21}\zeta_1^{x_2}\zeta_2^{x_1}$

For T_k, eigenvectors have the form (for $x_i \in [N]$)

$$f_{\zeta_1,\ldots,\zeta_k}(x_1, \ldots, x_k) = \sum_{\pi \in S_k} A_\pi \zeta_{\pi(1)}^{x_1} \ldots \zeta_{\pi(k)}^{x_k} = \det_A \begin{pmatrix} \zeta_1^{x_1} & \ldots & \zeta_k^{x_1} \\ \vdots & & \vdots \\ \zeta_1^{x_k} & \ldots & \zeta_k^{x_k} \end{pmatrix}$$

for some constants ζ_i and $A_\pi = A_\pi(\zeta_1, \ldots, \zeta_N)$

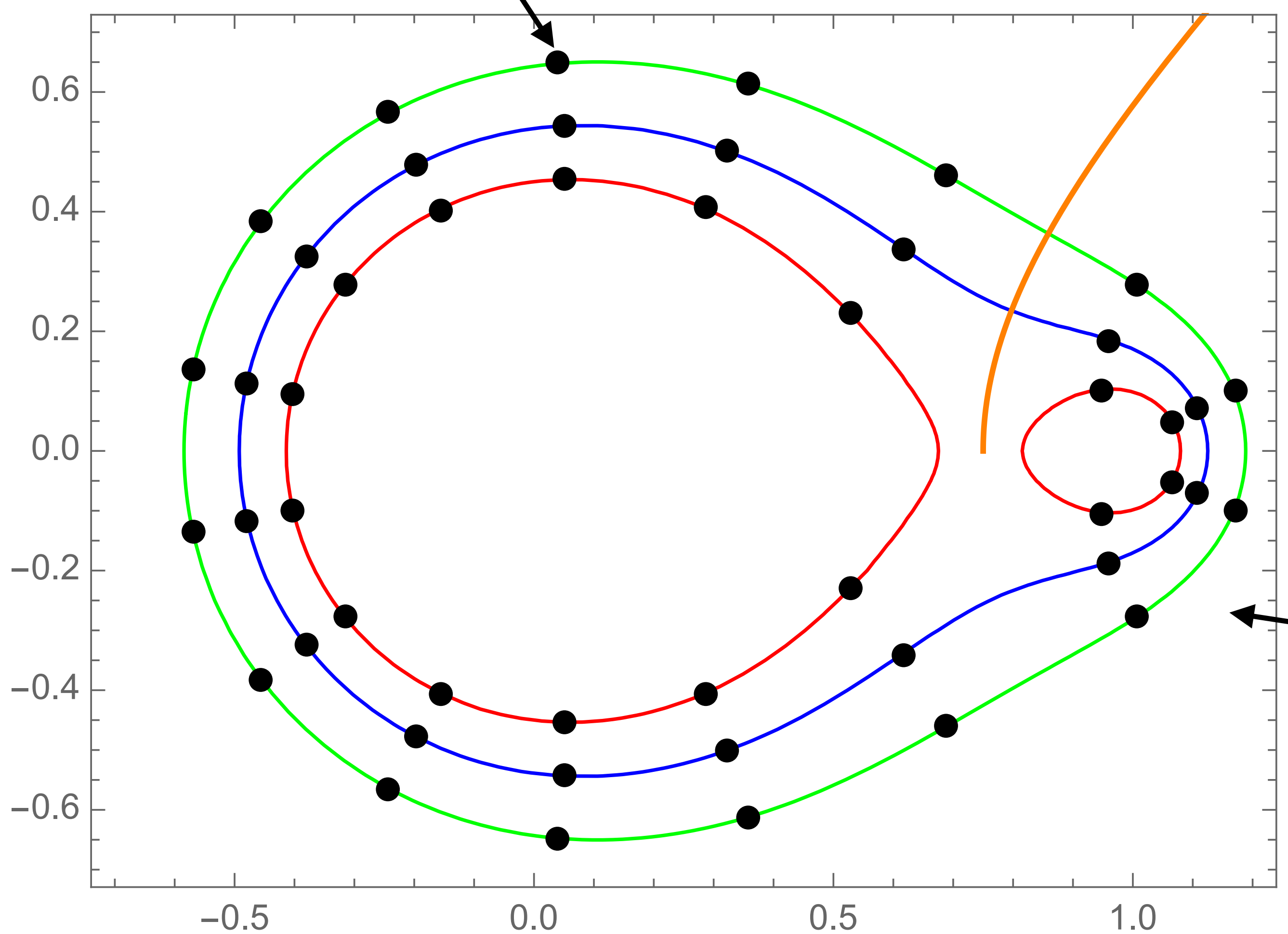
For T_n, the Bethe roots satisfy a system of polynomial equations
(Sutherland, Yang, Yang 1967)

$$\zeta_i^N = (-1)^{n-1} \prod_{j=1}^{n} \frac{1-(1-r^2)e^Y \zeta_j^{-1}}{1-(1-r^2)e^Y \zeta_i^{-1}},$$

Let $(1-r^2)e^Y w_j = \zeta_j$.

$$w_i^{N-n}(1-w_i)^n = -C \underbrace{\prod_{j=1}^{n} \frac{w_j - 1}{w_j}}_{\text{symmetric in all } w_j\text{s}}$$

$$w^{12}(1-w)^4 = const$$



3 ovals with the same n/N.

The rescaled Bethe roots w_i lie on "Cassini ovals"

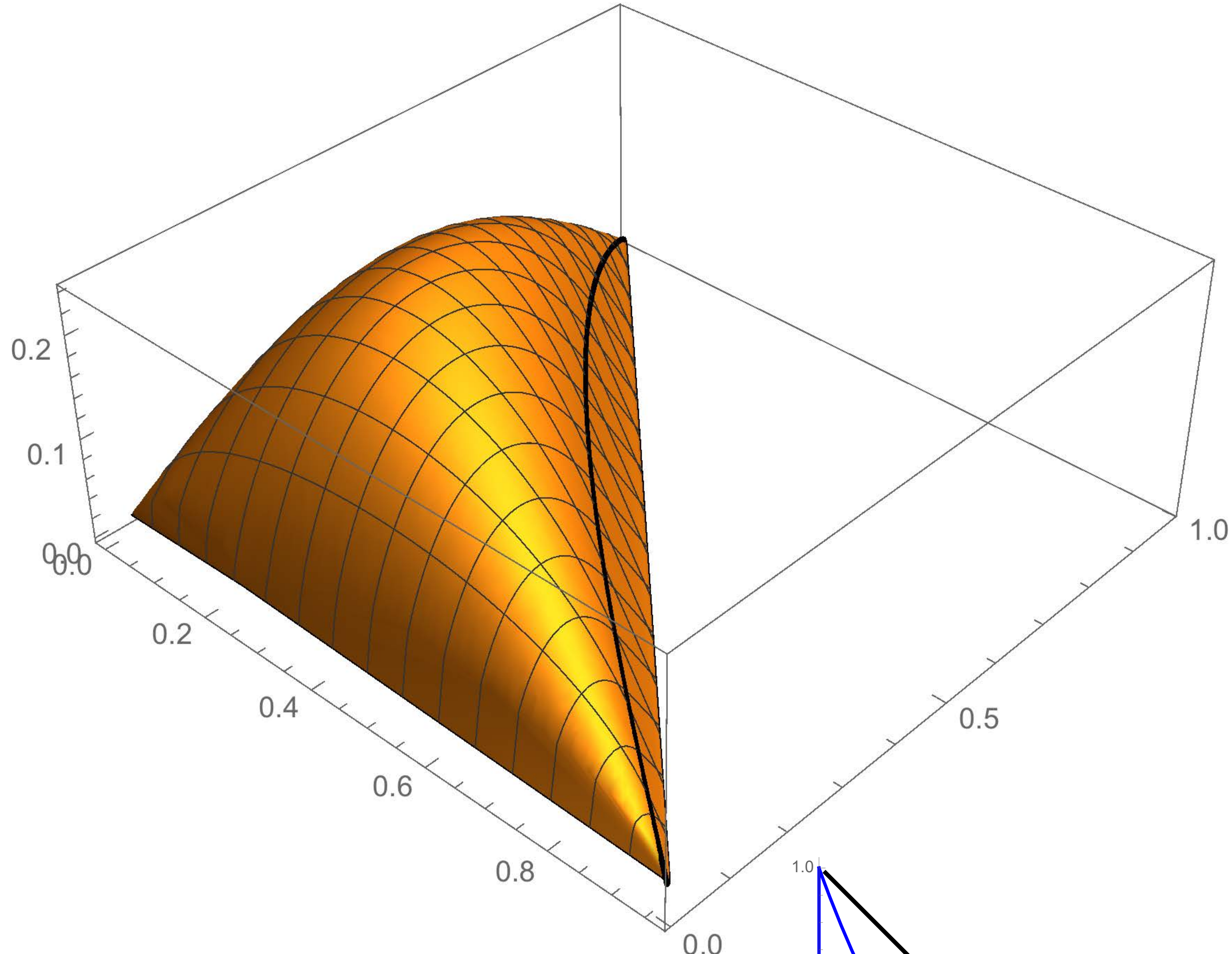
$$C_{\alpha,\beta} = \{w \ : \ \alpha \log |w| + \beta \log |1-w| = 1\}$$

The leading eigenvalue is

$$\Lambda = e^{Xn} e^{Y(N-n)} \left[1 - (-1)^n A r^{-2n} (1 - r^2)^N\right] \prod_{j=1}^{n} \frac{r^2}{1 - (1 - r^2) w_j}$$

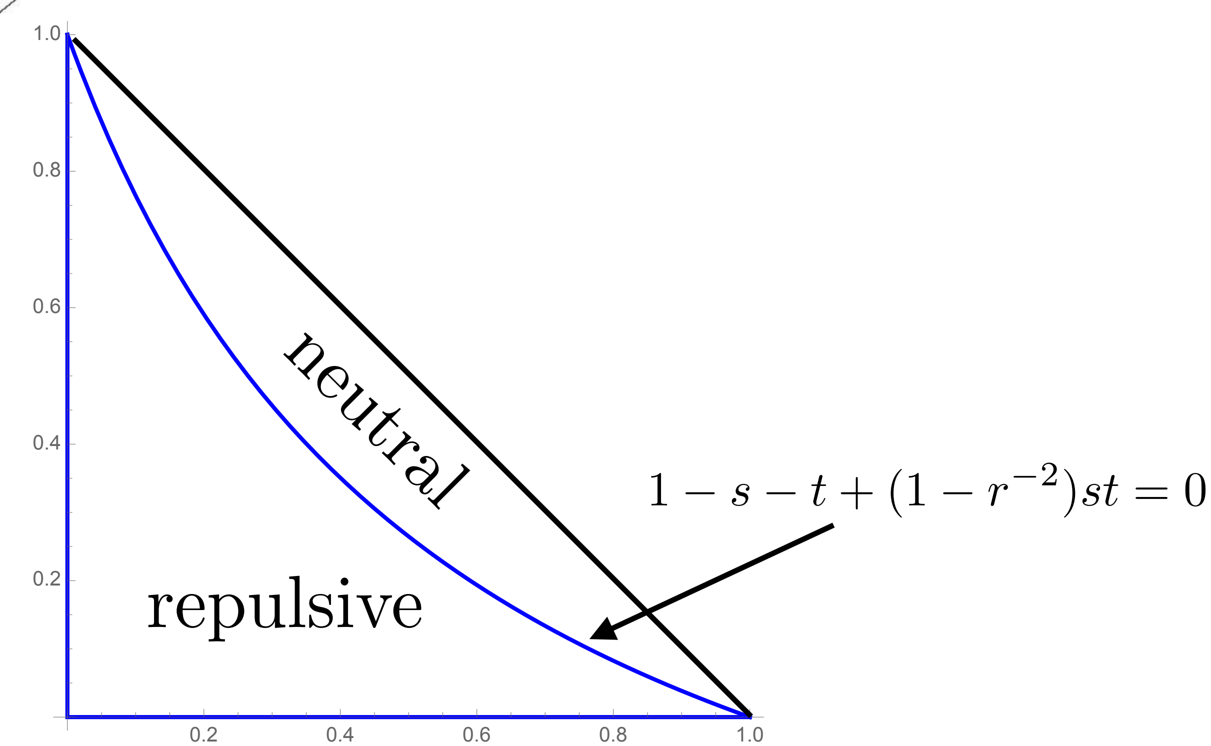
where $w_i^{N-n}(1 - w_i)^n = -A$

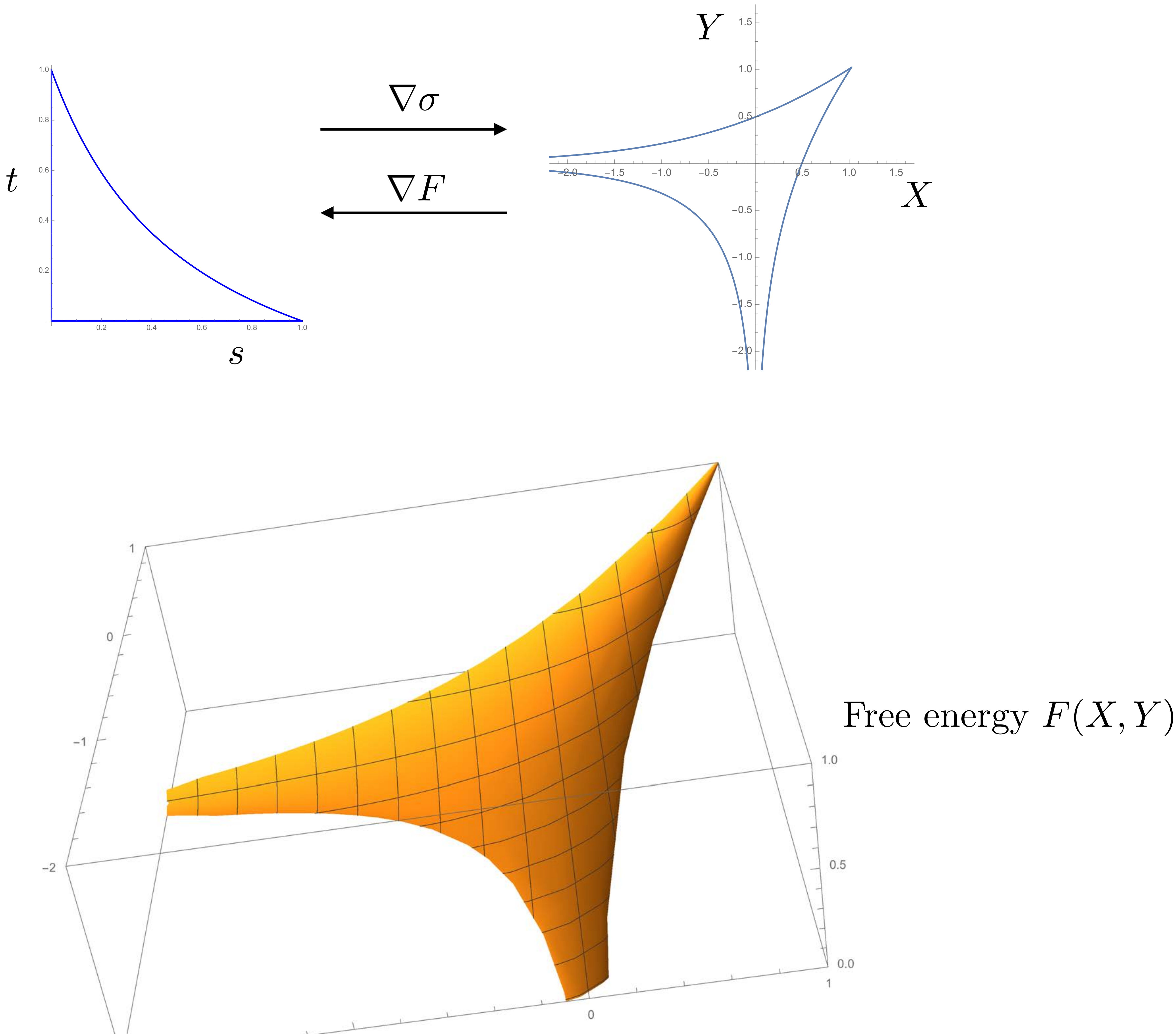
There is an explicit formula for $\sigma_r(s,t)$ in terms of the dilogarithm.



plot of $-\sigma$

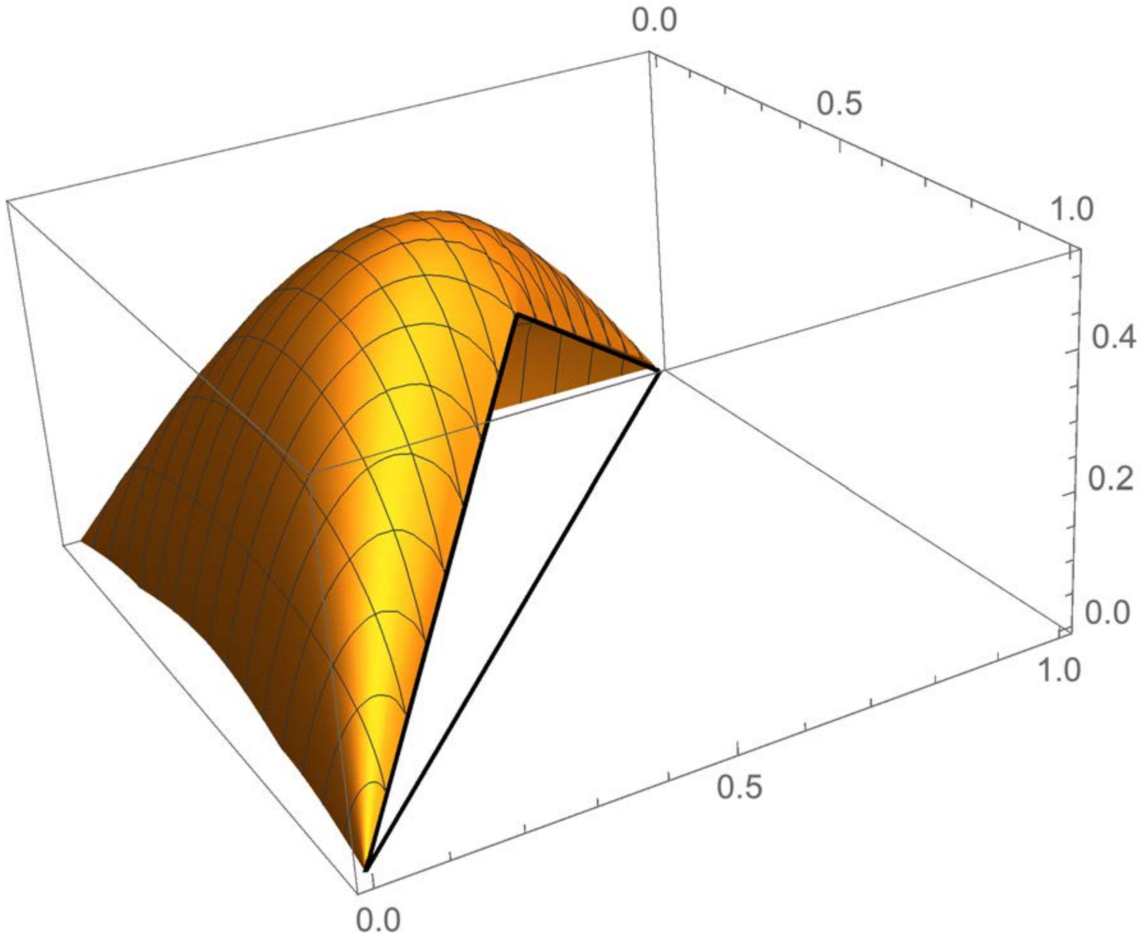
when $r < 1$, $\sigma(s,t)$ is piecewise analytic:

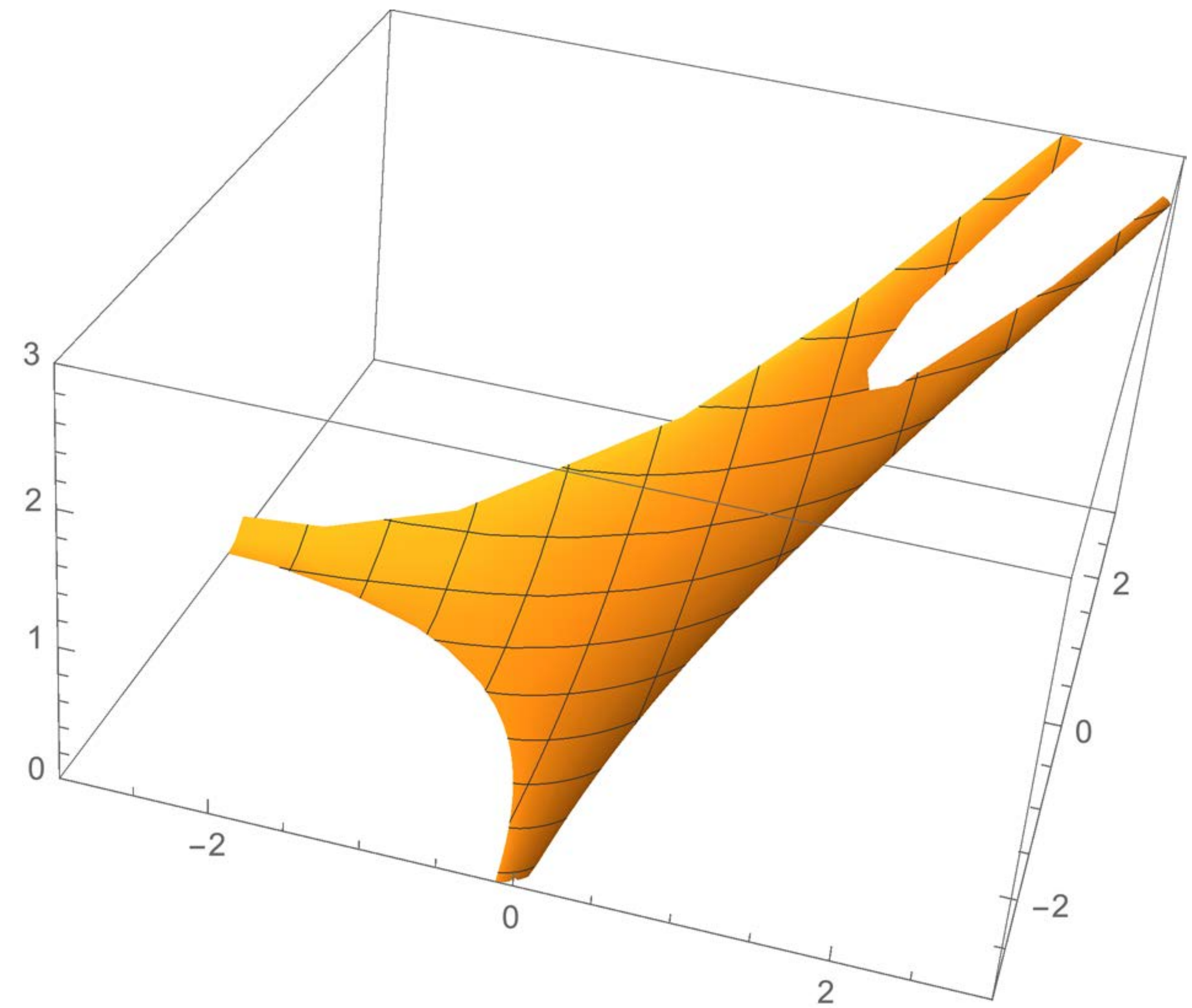




t
s
$\nabla\sigma$
∇F
Y
X
Free energy $F(X,Y)$

$r > 1$ case





σ F

no “neutral” phase

Thm: (...analogous limit shape theorem for 5-vertex model)

Cor: The PDE for the limit shape can be reduced to the PDE for $z = z(x, y)$:

$$z_x + f(z)z_y = 0$$

where $f(z)$ is explicit (but only real analytic, not complex analytic.)

Equivalently, we have a PDE for $x = x(z), y = y(z)$:

$$y_{\bar{z}} - f(z)x_{\bar{z}} = 0$$

The PDE for $x = x(u), y = y(u)$ (with $u = 1 - (1 - r^2)z$) is

$$A(\frac{1-u}{1-r^2})x_{\bar{u}} - A(\frac{u-r^2}{u(1-r^2)})y_{\bar{u}} = 0$$

where

$$A(z) = -z \arg z - (1-z) \arg(1-z)$$

One can still parametrize solutions with analytic functions...

Thm: The following universal formula holds:

$$zP_z x + wP_w y + h(x, y) + f(z) = 0.$$

Here $P(z, w) = 0$ is the spectral curve

$$P(z, w) = 1 - z - w + (1 - r^2)zw,$$

h is the height function and f is an arbitrary analytic function.

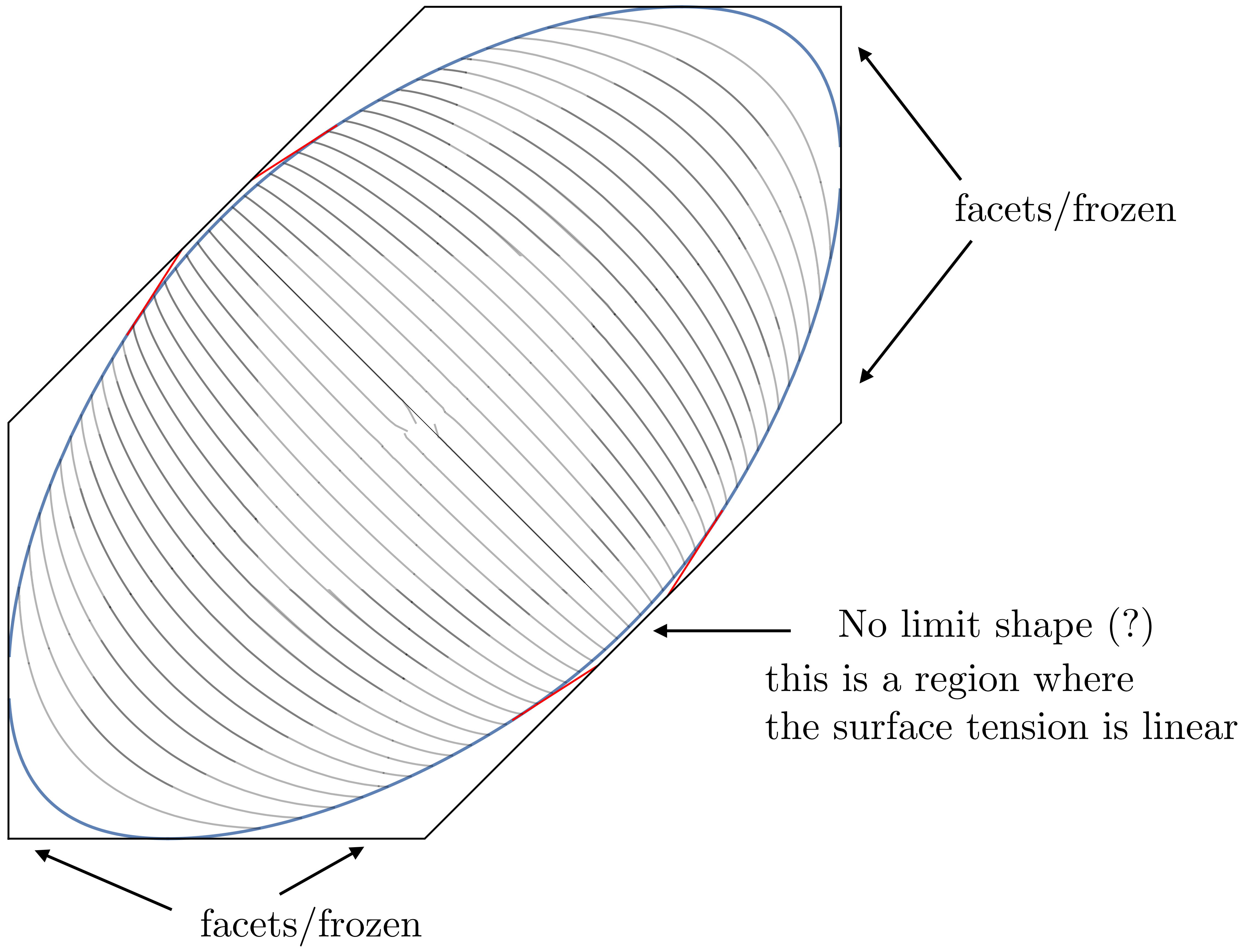
One can still parametrize solutions with analytic functions...

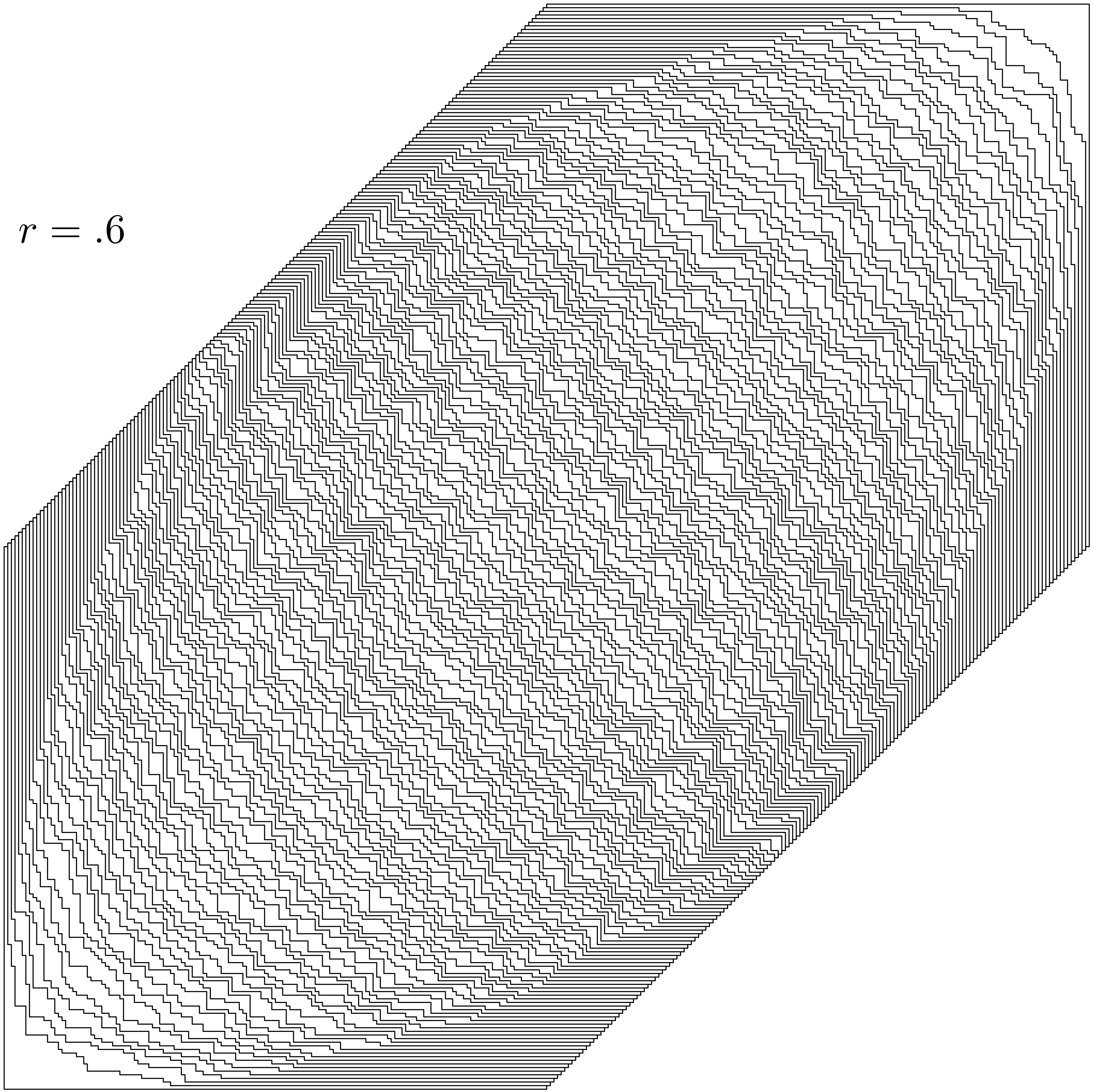
Set

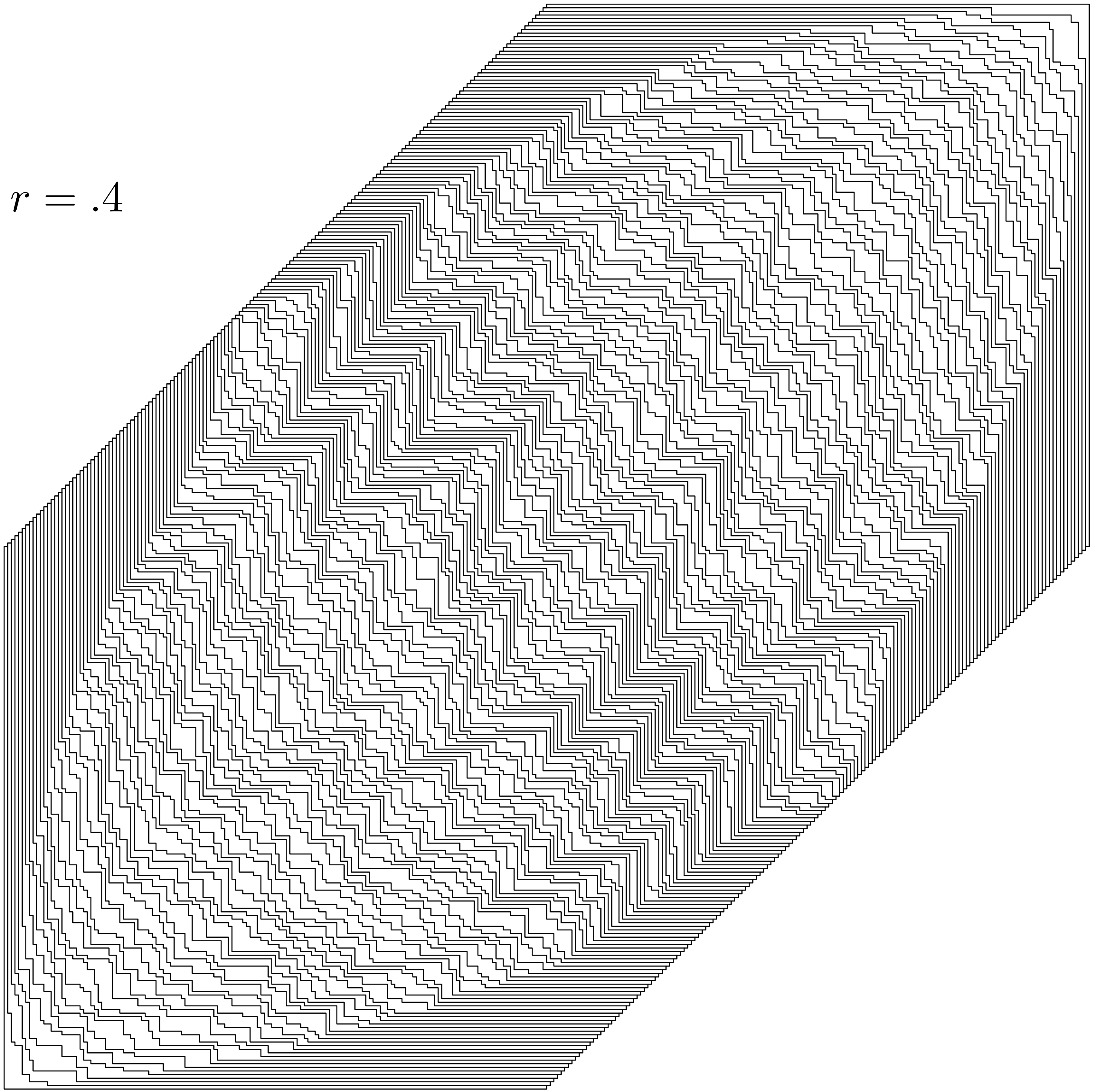
$$y = \frac{|u|^2}{r^2}\left(x - \frac{\mathrm{Im}(g(u))}{\mathrm{Im}(u)}\right)$$

where g is an arbitrary analytic function. Plug in to the PDE and integrate:

$$x = \frac{-1}{r^2 A(w) - |u|^2 A(z)}\mathrm{Im}\left(\int \frac{r^2 g(u)}{(1-u)(u-r^2)}du + \frac{|u^2|g(u)A(z)}{\mathrm{Im}(u)}\right).$$

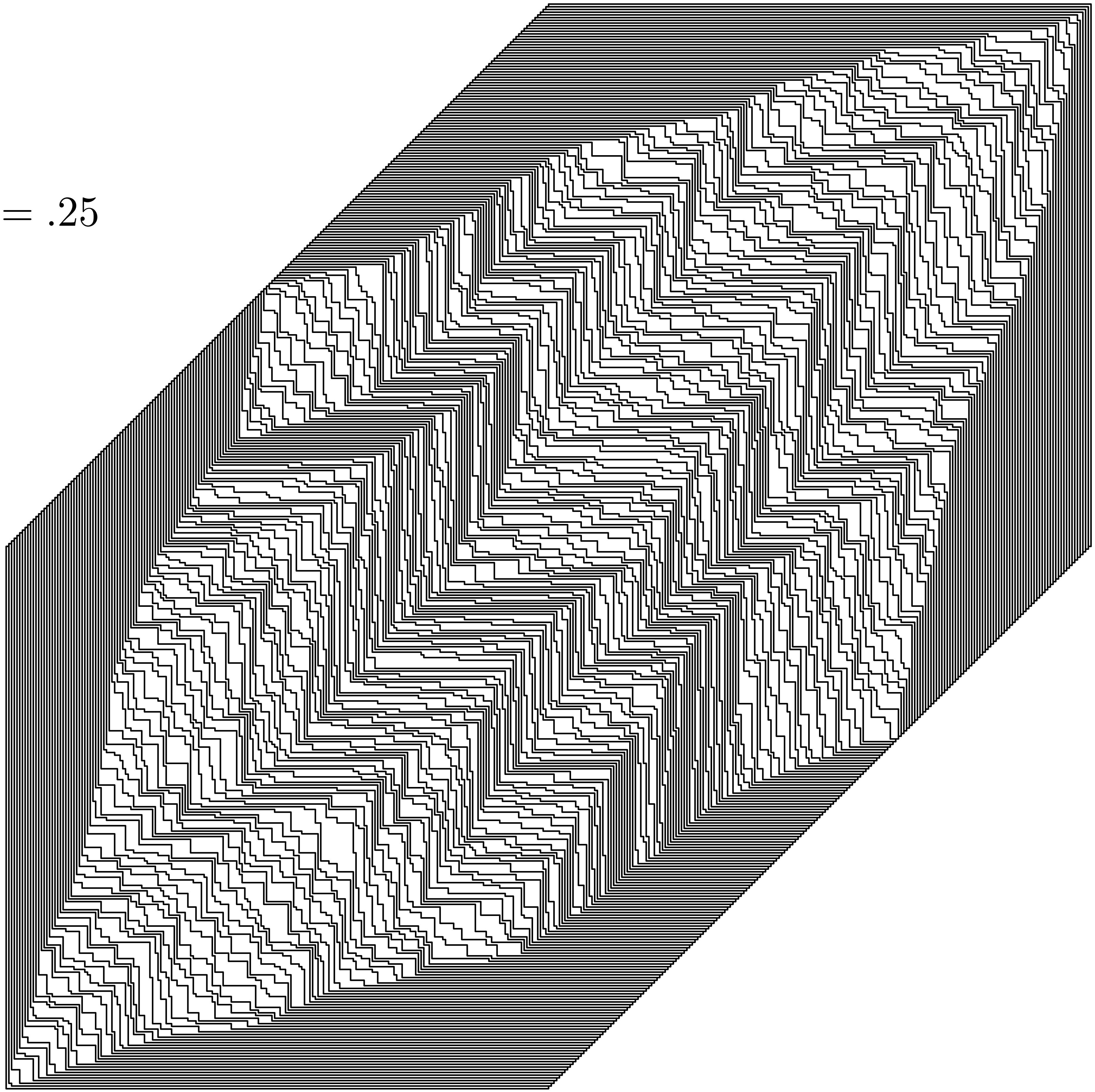

facets/frozen
No limit shape (?)
this is a region where
the surface tension is linear
facets/frozen

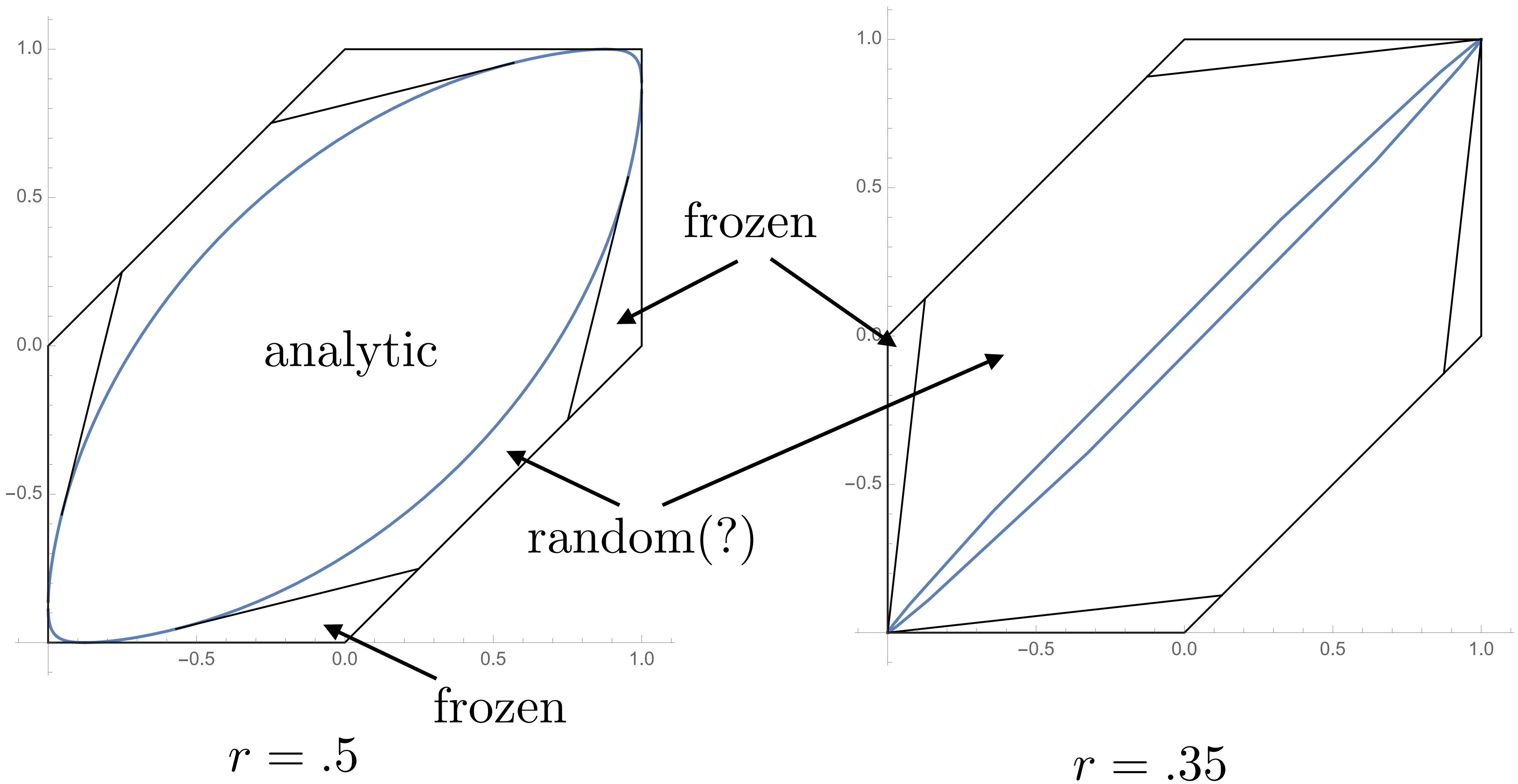

$r = .6$



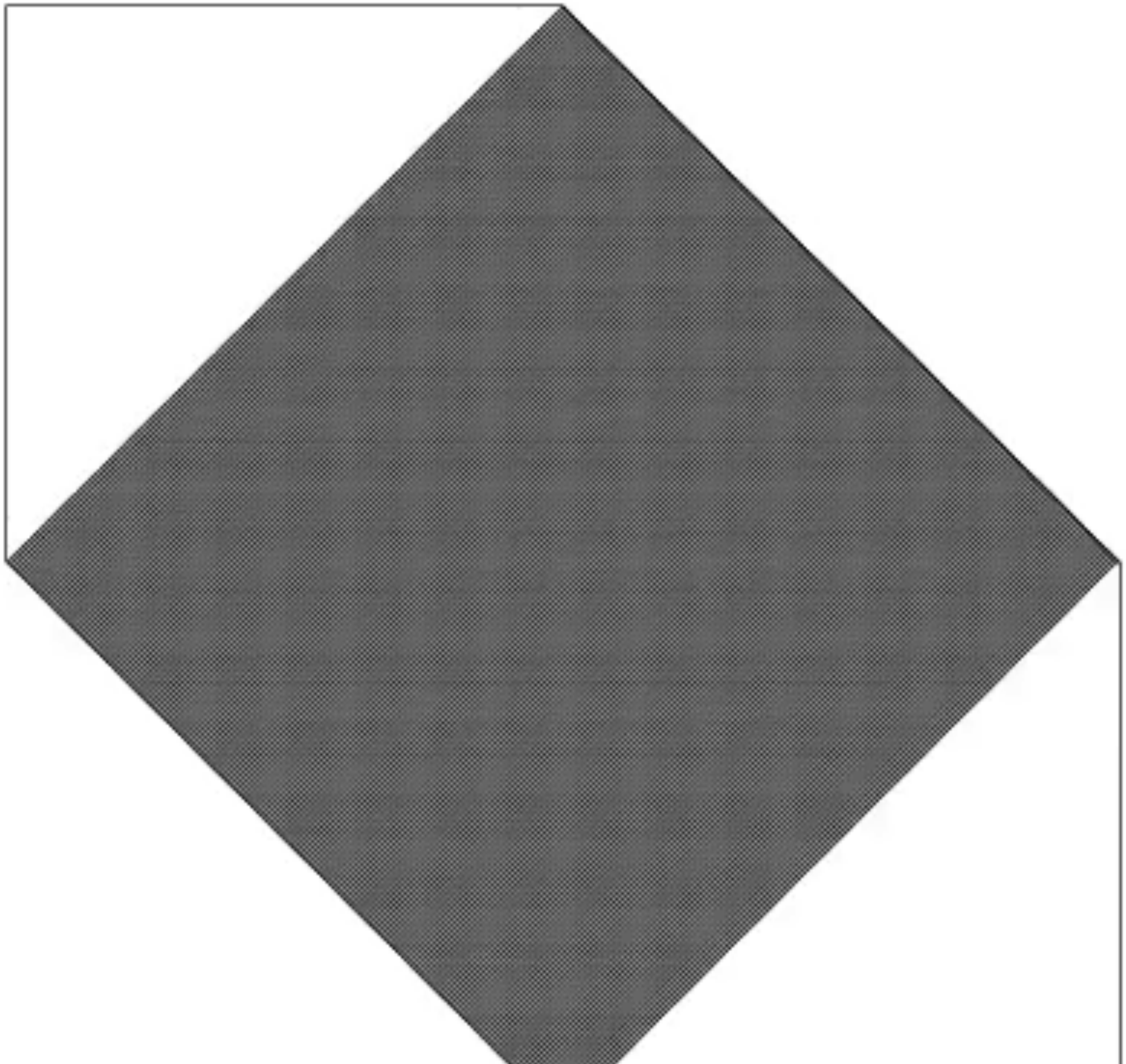
$r = .4$

$r = .25$





when $r \leq 1/3$ there is no limit shape.



Conjecture: Limit shapes for the 6-vertex model satisfy

$$zP_z x + wP_w y + f(z) + h = 0$$

where $P(z, w) = 0$ is the spectral curve and f is analytic, depending on the boundary

The evidence for this conjecture is that it is true on two different codim-1 subvarieties of the parameter space: the free fermionic subvariety and the 5-vertex subvariety.

THANK YOU

Fluctuations

The leading eigenvector of T_k has the form

$$f_{\zeta_1,\ldots,\zeta_k}(x_1,\ldots,x_k) = \sum_{\pi\in S_k} A_\pi \zeta_{\pi(1)}^{x_1}\ldots\zeta_{\pi(k)}^{x_k} = \det{}_A \begin{pmatrix} \zeta_1^{x_1} & \ldots & \zeta_k^{x_1} \\ \vdots & & \vdots \\ \zeta_1^{x_k} & \ldots & \zeta_k^{x_k} \end{pmatrix}$$

and

$$\Pr(\text{particles at } x_1, x_2, \ldots, x_k) \propto f(x_1,\ldots,x_k) f(N-x_1,\ldots,N-x_k)$$

in our case

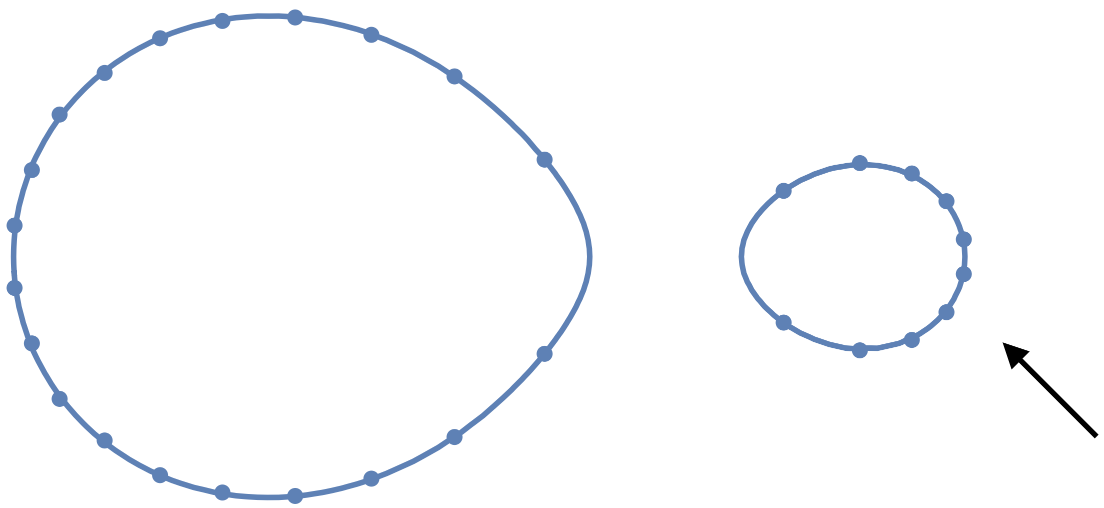
$$A_\pi = (-1)^\pi \prod_{1\le i<j\le n} (1-\zeta_{\pi(j)}^{-1})$$

leading to

$$f = Q^{x_1+\cdots+x_n} \det \begin{pmatrix} \zeta_1^{x_1} & \ldots & \zeta_n^{x_1} \\ (1-\zeta_1^{-1})\zeta_1^{x_2} & \ldots & (1-\zeta_n^{-1})\zeta_n^{x_2} \\ \vdots & & \vdots \\ (1-\zeta_1^{-1})^{n-1}\zeta_1^{x_n} & \ldots & (1-\zeta_n^{-1})^{n-1}\zeta_n^{x_n} \end{pmatrix}$$

with $Q = (\prod_{j=1}^n \zeta_j)^{-1/n}$.

We can compute $f(1, 2, 3, \ldots, k)$ and $f(N/k, 2N/k, \ldots, (k-1)N/k)$; in the attracting phase their ratio tends to 1 as $N, k \to \infty$.



These 10 particles are distributed like electrons in a conductor

Thm: In the attracting phase particles on a given row or column are uniformly located (conditionally on being disjoint). Height fluctuations are $O(\sqrt{N})$.