Quantitative stochastic homogenization

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July 27, 2018

Elliptic equations

We consider

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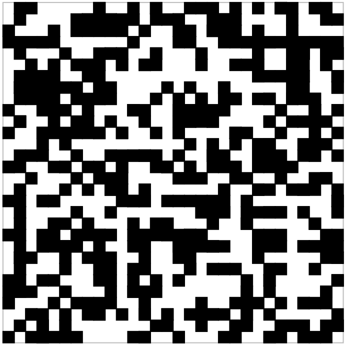
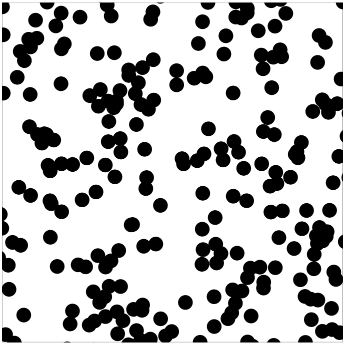
finite range of dependence

Coefficients





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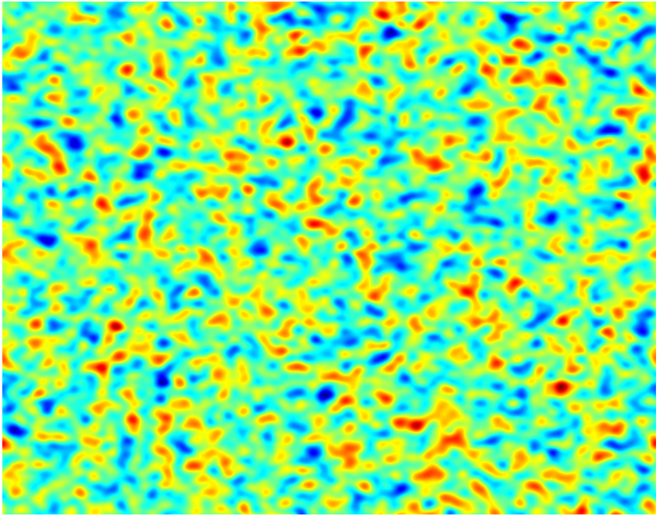
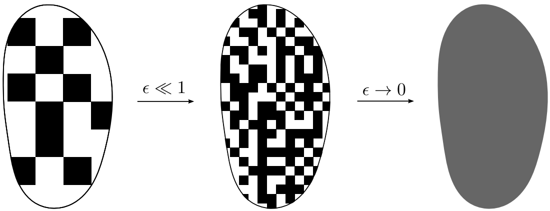


Scaling

$$\begin{cases} -\nabla \cdot (\mathbf{a}(\varepsilon^{-1}\cdot)\nabla u_\varepsilon) = 0 & \text{in } U, \\ u_\varepsilon = f & \text{on } \partial U. \end{cases}$$



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There exists a matrix $\bar{\mathbf{a}}$ s.t.

$$u_\varepsilon \xrightarrow[\varepsilon \to 0]{L^2} \bar{u},$$

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$$\nabla u_\varepsilon \rightharpoonup \nabla \bar{u}, \qquad \mathbf{a}(\varepsilon^{-1}\cdot)\nabla u_\varepsilon \rightharpoonup \bar{\mathbf{a}}\nabla \bar{u}.$$

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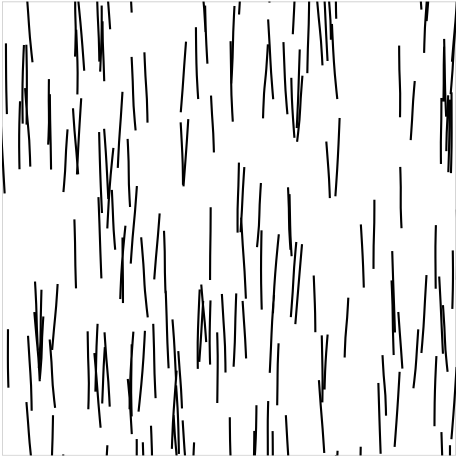

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Goal: estimate rates of convergence.

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- 1st approach (Gloria, Neukamm, Otto, . . .): "non-linear" concentration inequalities (cf. also Naddaf-Spencer).
- 2nd approach (Armstrong, Kuusi, M., Smart, . . .): renormalization, focus on energy quantities.

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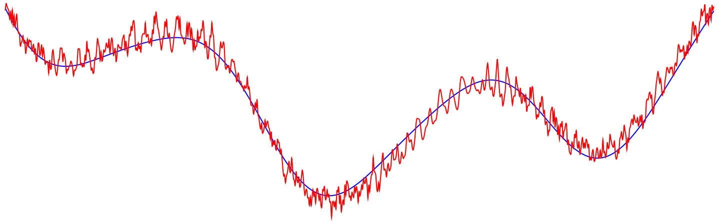
- Renormalization: very inspiring, broad and powerful idea, with still a lot of potential as a mathematical technique
- Develop tools that will hopefully shed light on variety of other problems: other equations, Gibbs measures, interacting particle systems, etc.
- Suggests new numerical algorithms

Problem reduction



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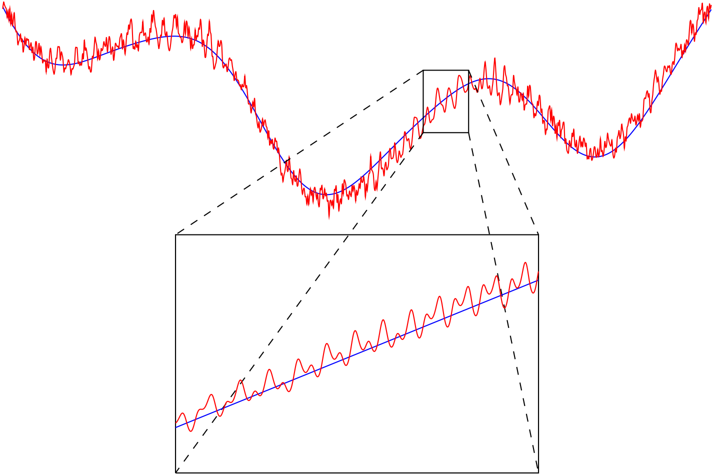

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For $p \in \mathbb{R}^d$, write $\mathbf{a}$-harmonic function with slope p as

$$x \mapsto p \cdot x + \phi_p(x),$$

that is,

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Quantify

$$\text{Spat. av. } \nabla \phi_p \to 0$$
$$\text{Spat. av. } \mathbf{a}(p + \nabla \phi_p) \to \bar{\mathbf{a}} p.$$

Gradual homogenization

$$\text{If} \qquad 1 - \delta \leqslant \mathbf{a}(x) \leqslant 1 + \delta,$$

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Linearization for $r \gg 1$.

Energies

Dal Maso, Modica 1986:

$$\nu(U,p) := \inf_{v \in \ell_p + H_0^1(U)} \frac{1}{2} \fint_U \nabla v \cdot \mathbf{a} \nabla v.$$

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$$\nu(\square, p) \xrightarrow[|\square| \to \infty]{\text{a.s.}} \frac{1}{2} p \cdot \bar{\mathbf{a}} p.$$

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- $\Longrightarrow$ Exponent improvement $\alpha \to (2\alpha) \wedge \frac{1}{2}$.

Corrector estimates

For $\square_r := \left]-\frac{r}{2}; \frac{r}{2}\right[^d$,

$$\mathbf{a}(x + \square_r) \simeq \bar{\mathbf{a}} + W_r(x), \qquad \text{where} \quad W_r(x) \simeq \mathcal{N}(0, r^{-d}).$$

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$$-\nabla \cdot \left(\bar{\mathbf{a}} + W_r(x)\right) \left(p + \nabla \phi_{p,r}\right) \simeq 0.$$

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$$\implies \qquad r^{\frac{d}{2}} (\nabla \phi_p)(r\ \cdot) \xrightarrow[r \to \infty]{\text{law}} \nabla (GFF).$$

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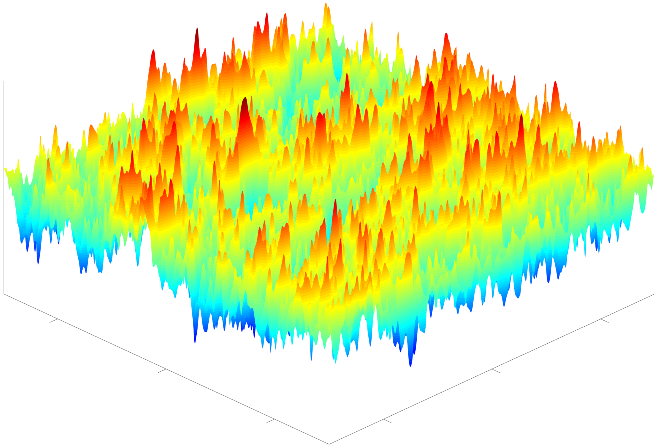

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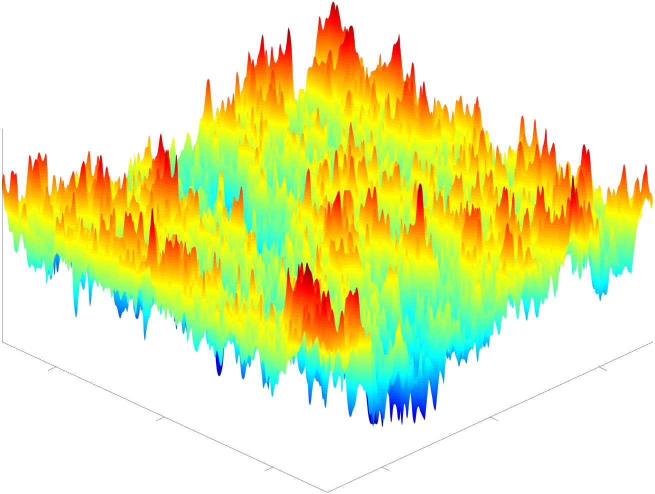
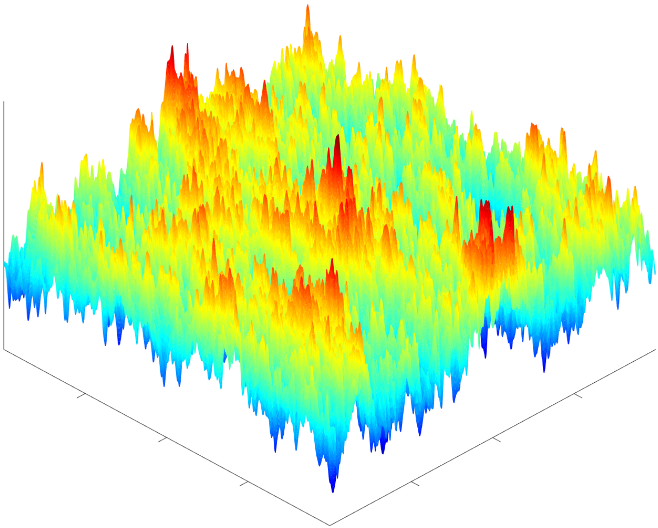


GFF $-\nabla \cdot \bar{a} \nabla \Phi = \nabla \cdot W$



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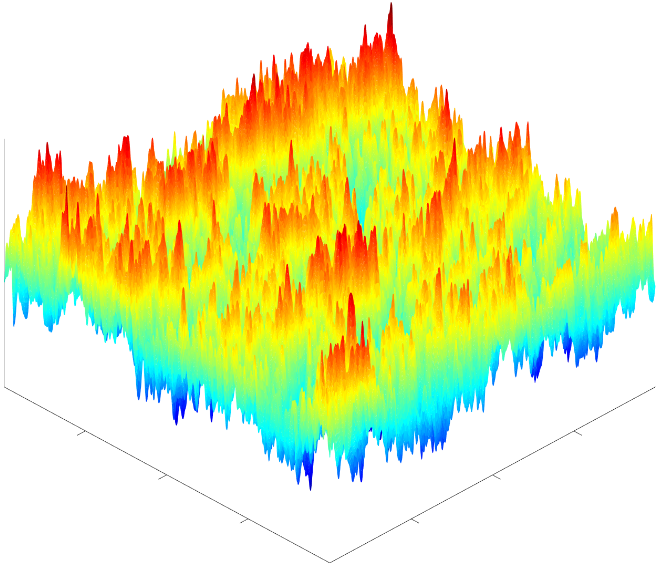

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Thank you!