ICMP 2018, Montreal

Conformal Field Theory and Critical Phenomena in $D=3$

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$$\begin{tikzcd}[column sep=small] & \text{Critical Phenomena} \arrow[dl] \arrow[dr] & \\ \text{CFT} & & \text{RG} \end{tikzcd}$$

complementary

- in 2d, since 1980's
- in $d \geqslant 3$, last ~10 years 'Conformal bootstrap revival'

Review: Poland, Rychkov, Vichi 1805.04405

CONF. INVARIANCE

OPERATOR PRODUCT EXPANSION (OPE)

REFLECTION POSITIVITY (UNITARITY)

CONF. BOOTSTRAP

$\Rightarrow$ Mathematically well-defined equations on critical point parameters
(CFT data = critical exponents + more)

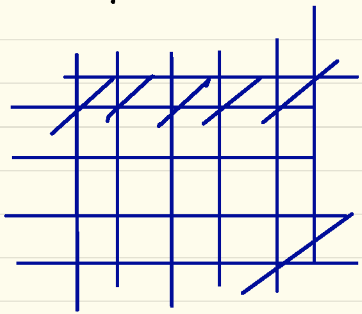
Plan:
1. How to derive these eqns
2. How to analyze them
3. Some results

BASIC CFT in $d \geq 3$

- CFT = critical point \ lattice artifacts

<u>Example:</u>

3d Ising
$T = T_c$



$s_i = \pm 1$
cubic lattice

$$\langle s(0)\, s(r) \rangle \sim \sum_{i=1}^{\infty} \frac{c_i}{r^{\gamma_i}} \qquad (r \gg a)$$

+ small rotation-inv breaking

- **Junk:** •) $r \approx a$
  •) constants c_i

Universal: γ_i

- In CFT we introduce 'local operators'

normalization

$$\langle \sigma_i(0)\, \sigma_i(r) \rangle = \frac{1}{r^{\gamma_i}} \qquad \gamma_i = 2\Delta_i$$

at all distances!

scale inv...

[One can think of σ_i as linear combinations of lattice operators

$$S + "S^3" + "S^5" + \dots$$

But we don't need to know this when using CFT]

WHY SO MANY OPERATORS?

- In 2d Ising : $\mathbb{1}$, σ, ε

- In 3d Ising
  - $\mathbb{1}$
  - σ_1, σ_2, σ_3, ...
  - ϵ_1, ε_2, ε_3, ...
  - Tensor operators (non-derivative)

<u>Answer</u> : σ_i exist also in 2d
but related to $\sigma = \sigma_1$

$$\Delta(\sigma_i) - \Delta(\sigma) \in \mathbb{N}$$

- In 3d

$$\Delta(\sigma) = 0.5181489(10) \iff$$

$$\Delta(\sigma_2) = 5.2906(11)$$

$\boxed{\begin{array}{l} \eta \quad \text{crit. exponent} \\ \eta = 2\Delta(\sigma) - 1 \end{array}}$

<u>Traditionally</u>: Think of σ as 'fundamental' and $\sigma_2, \sigma_3, \ldots$ as 'composite'

(e.g. $\sigma \sim \varphi$, $\sigma_2 \sim \varphi^5$ in Landau-Ginzburg)

▶ CFT: All these operators are equally fundamental

(But decoupling – reduced sensitivity to higher-dimension operators)

Any CFT will contain ∞ many operators $\mathcal{O}_i(x)$ characterized by

$$\underbrace{\begin{array}{ll} - \ \Delta_i \in \mathbb{R} & \text{scaling dimension} \\ - \ \pi_i & SO(d) \text{ representation} \end{array}}_{\text{spectrum of the theory}}$$

(no fermions today)

▶ Basics of conf. transformations

Conf. group of $\mathbb{R}^d \cup \{\infty\}$

$$= SO(d+1,1)$$

$$f : x \mapsto x' \quad \text{s.t.} \quad \frac{\partial f^\mu}{\partial x^\nu} = \underbrace{\Omega(x)}_{>0} \underbrace{R^\mu{}_\nu(x)}_{\in SO(d)}$$

- Lie algebra generators

$P^\mu, M^{\mu\nu}, D, K^\mu \Leftrightarrow$ conformal Killing vector fields

e.g.

$$\mathcal{E}(D) = x^\mu \partial_\mu$$

$$\mathcal{E}(K^\mu) = 2x^\mu x^\nu \partial_\nu - x^2 \partial^\mu$$

- Conformal group action on $\mathcal{O}\,[\Delta, \pi]$

$$(\rho_f \circ \mathcal{O})(x') = \Omega(x)^{-\Delta}\, \pi\left[R^{\mu}{}_{\nu}(x)\right] \circ \mathcal{O}(x)$$

$$x \xrightarrow{f} x' \qquad \frac{\partial f^{\mu}}{\partial x^{\nu}} = \Omega(x)\, R^{\mu}{}_{\nu}(x)$$

[induced representation ρ from KDM subgroup preserving $x=0$]

- CONFORMAL INVARIANCE:

$$\langle O_1(x_1)\, O_2(x_2) \ldots O_n(x_n) \rangle \in \left(\rho_1 \otimes \rho_2 \otimes \ldots \otimes \rho_n\right)^{SO(d+1,1)}$$

- For 2pt functions

$\langle O_1(x_1)\, O_2(x_2)\rangle \neq 0$ only for $\Delta_1 = \Delta_2$
and $\rho_1 = \rho_2^{+}$
(dual reflected)

- Specialize to rank-ℓ tensors
- Assume spectrum nondegenerate

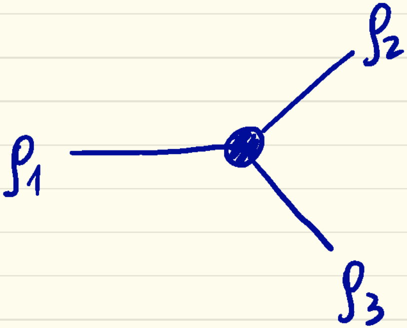
$\Rightarrow$ • $\mathcal{O}(x)$ only has 2pt with itself

normalization

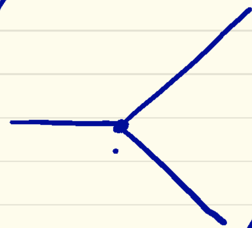
$$\langle O^{(\alpha)}(x)\, O^{(\beta)}(y)\rangle = \frac{1}{|x-y|^{2\Delta}}\, I_{\alpha,\beta}(x-y)$$

unique, know
tensor structure
for each ℓ

▶ For 3pt functions – finite-dim.



$$= \sum_{i=1}^{I} \lambda_i^{CFT} \Big(\quad \Big)_i$$



OPE coeffs.

$$\boxed{\text{CFT data} = \text{Spectrum} \cup \text{OPE coeffs}}$$

Simple rule to count 3pt functions:

– use conf. transformations to fix 3 points on a line

$$I = \dim\left(\pi_1 \otimes \pi_2 \otimes \pi_3\right)^{SO(d-1)}$$

E.g. (scalar – scalar – rank ℓ) gives $I = 1 \quad \forall\, \ell$

▶ For 4pt functions – ∞-dimensional

E.g. for scalars:

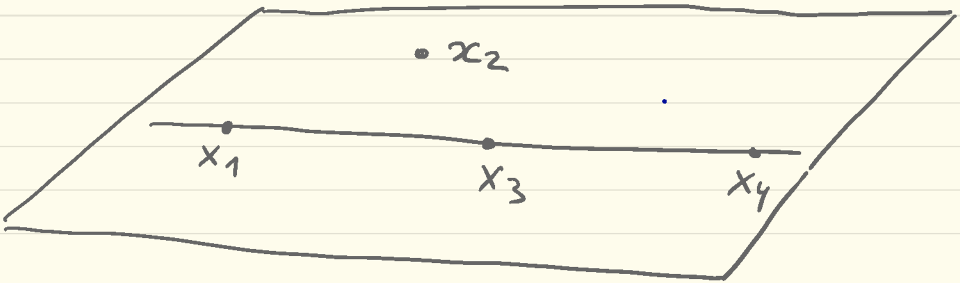
$$\langle \mathcal{O}_1(x_1)\,\mathcal{O}_2(x_2)\,\mathcal{O}_3(x_3)\,\mathcal{O}_4(x_4)\rangle = K \cdot g(u,v)$$

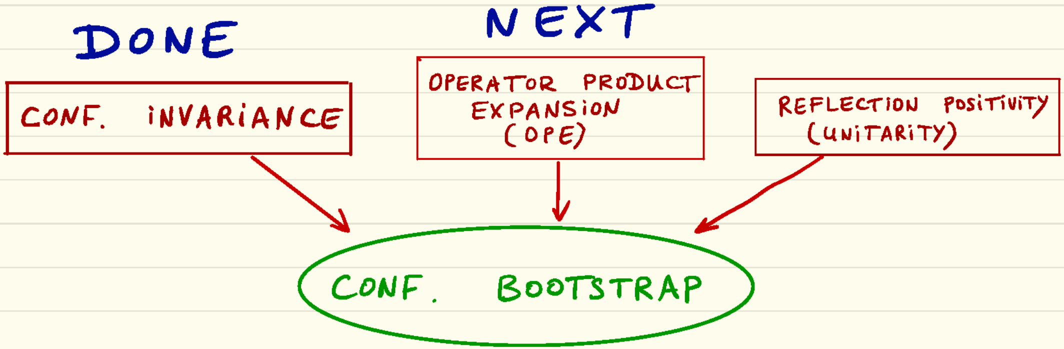
$K = K(x_i \,|\, \Delta_i)$ – fixed

$g(u,v)$ arbitrary

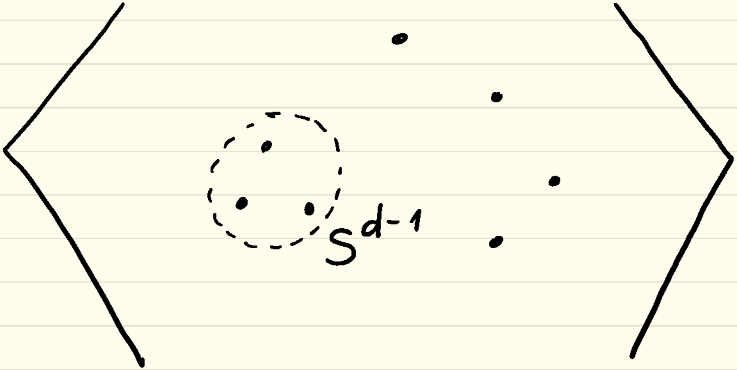
$$u = \frac{x_{12}^2\; x_{34}^2}{x_{13}^2\; x_{24}^2}\,, \quad v = u\big|_{1\leftrightarrow 3}$$

conf. inv. cross-ratios




DONE
NEXT
CONF. INVARIANCE
OPERATOR PRODUCT EXPANSION (OPE)
REFLECTION POSITIVITY (UNITARITY)
CONF. BOOTSTRAP

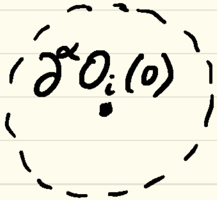
OPE = COMPLETENESS RELATION



State $|\Psi\rangle$ in Hilbert space on S^{d-1}

Full correlation function is scalar product $\langle \Psi_1 | \Psi_2 \rangle$

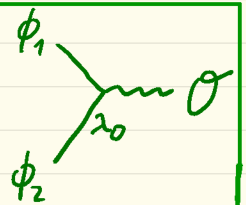
- Local operators at the center generate many states:



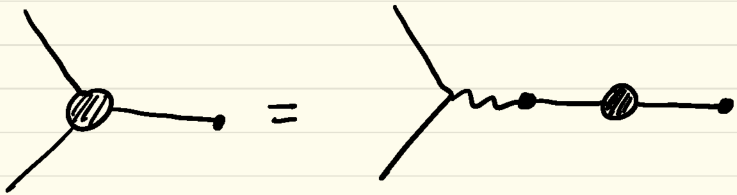
- Assumption: these states are complete ('state-operator correspondence')

$$\phi_1(x_1)\,\phi_2(x_2) = \sum_{\mathcal{O}} \lambda^{CFT}_{\mathcal{O}}\, C^{\alpha}_{\mathcal{O}}(x_1,x_2)\,\partial^{\alpha}\mathcal{O}(0)$$

OPE:



Consistency:



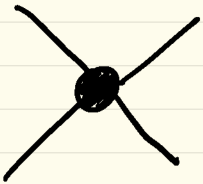
$\Rightarrow$ same $\lambda_{\mathcal{O}}$ as in 3pt function

- OPE must be consistent with conformal invariance

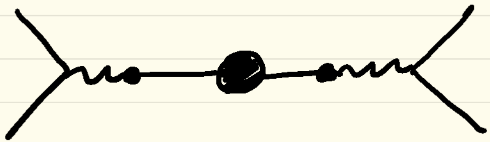
$C^{\alpha}_{\mathcal{O}}(x_1, x_2)$ are Clebsch-Gordan coefficients between $\rho_1 \otimes \rho_2$ and representation $\{\partial^{\alpha}\mathcal{O}(0)\} \simeq \rho_{\mathcal{O}}$

NB. OPE $\neq$ tensor product decomposition

▶ OPE for 4pt functions



$$= \sum_{\mathcal{O}} \lambda_{\mathcal{O}}^2$$

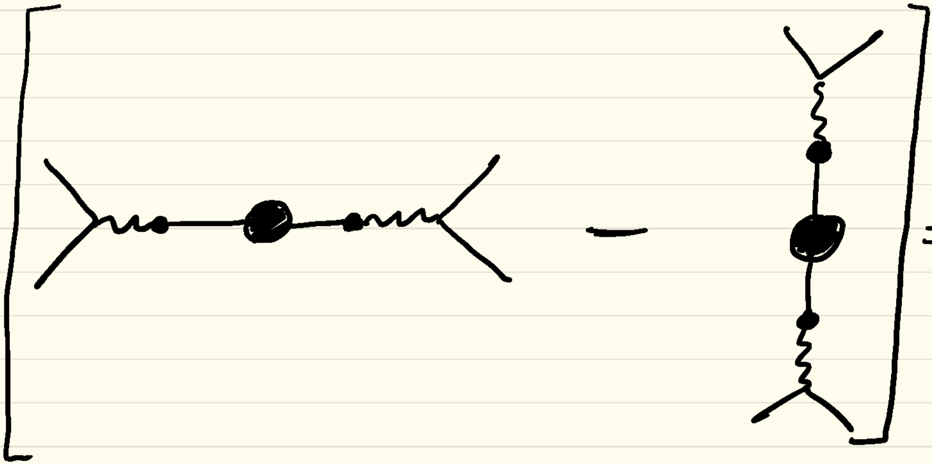


conformal block

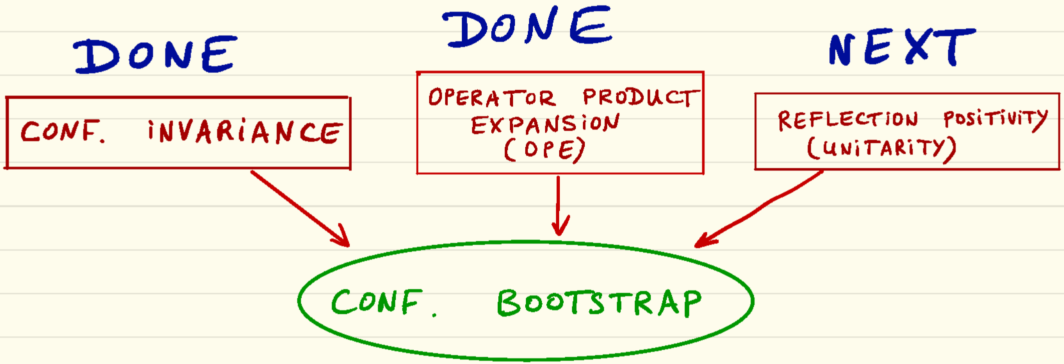
(∞-dim space of invariant 4pt functions)

MAIN EQN. OF CONF. BOOTSTRAP

$$\sum_{\mathcal{O}} \lambda_{\mathcal{O}}^2$$



$$= 0$$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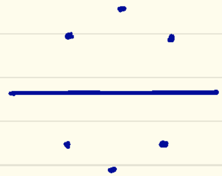
CONF. INVARIANCE
OPERATOR PRODUCT EXPANSION (OPE)
REFLECTION POSITIVITY (UNITARITY)
CONF. BOOTSTRAP

REFLECTION POSITIVITY

$$\langle F \; \Theta(F) \rangle \geq 0$$



$\Rightarrow$ 2 types of constraints

- lower bounds on operator dimensions

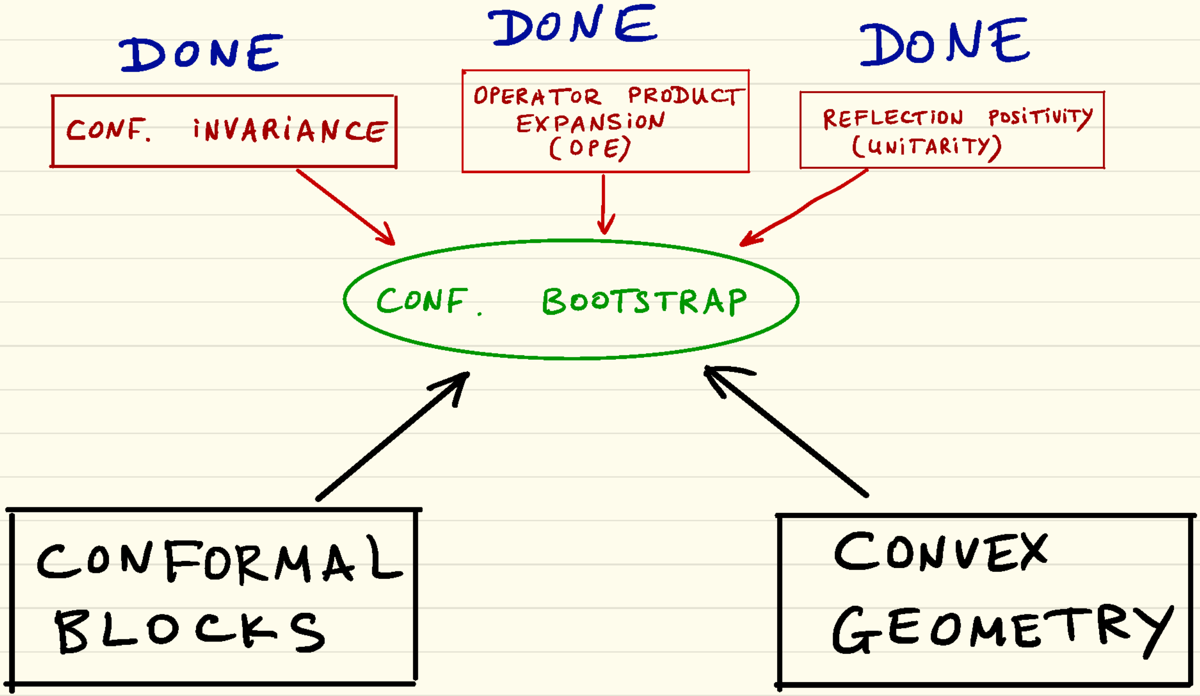
  In 3d $\quad \Delta(\text{scalars}) \geq 1/2$

  $\qquad\qquad \Delta(\text{rank } \ell) \geq \ell + 1$

- reality constraints on OPE coeffs

  $$\lambda_{\mathcal{O}} \in \mathbb{R}$$

[Kind of obvious in 3d Ising since everything is real]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CONF. INVARIANCE
OPERATOR PRODUCT EXPANSION (OPE)
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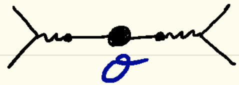
CONF. BOOTSTRAP
CONFORMAL BLOCKS
CONVEX GEOMETRY

CONFORMAL BLOCKS



$$K(x_i \mid \Delta_i)\; G_{\Delta,\ell}(u,v)$$

dimension and rank of 'exchanged' $\mathcal{O}$

- Satisfy 2nd order PDE
  ↳ connections to integrability [Isachenkov, Schomerus 1602.01858]

- For d even, explicit ${}_2F_1$ solutions

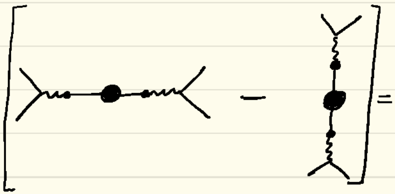
- For d odd, fast convergent power-series expansion exploiting meromorphic structure (poles at Δ outside physical region)

- In practice we compute them numerically with arbitrary precision

CONVEX GEOMETRY

CONSIDER BOOTSTRAP EQN FOR 4 IDENTICAL SCALARS

OPE: $\phi \times \phi = \mathbb{1} + \sum \lambda_\sigma \sigma$

$\sum_\sigma \lambda_\sigma^2$  $= 0$

$$\underbrace{\sum_{\sigma \neq 1} \lambda_\sigma^2 F_\sigma(u,v)} = \underbrace{u^{\Delta_\phi} - v^{\Delta_\phi}}_{\text{unit operator}}$$

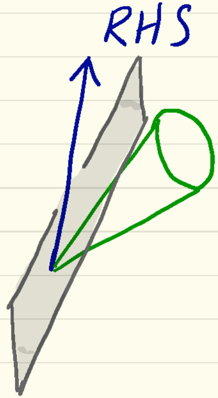
$\lambda_1 = 1$

For fixed spectrum and varying $\lambda_\sigma^2 \geq 0$

LHS fills a <u>convex cone</u>

solution exists

no solution



can prove this by exhibiting 'separating hyperplane'
= linear functional α s.t.

$$\alpha(RHS) < 0$$

$$\alpha\left(F_\theta(u,v)\right) \geq 0 \qquad \forall\, \theta \in Spec.$$

CONVEX GEOMETRY FOR MULTIPLE CORRELATORS

$$\langle \phi_i \, \phi_j \, \phi_k \, \phi_\ell \rangle \qquad i,j,k,\ell = 1 \ldots N$$

Will involve products $\lambda_{ij\mathcal{O}} \, \lambda_{k\ell\mathcal{O}} =: M^{\mathcal{O}}_{AB}$

$A = (ij)$

$B = (k\ell)$

NB: $M^{\mathcal{O}} \succcurlyeq 0$ (positive semidefinite)

M enters linearly in bootstrap eqns:

$$\underbrace{\sum_{\mathcal{O}} \mathcal{L}_{\mathcal{O}}(M^{\mathcal{O}})}_{\text{convex cone}} = RHS \in \mathbb{R}^{N^4} \otimes f(u,v)$$

explicit linear operator involving various conf. blocks

Separating plane conditions:

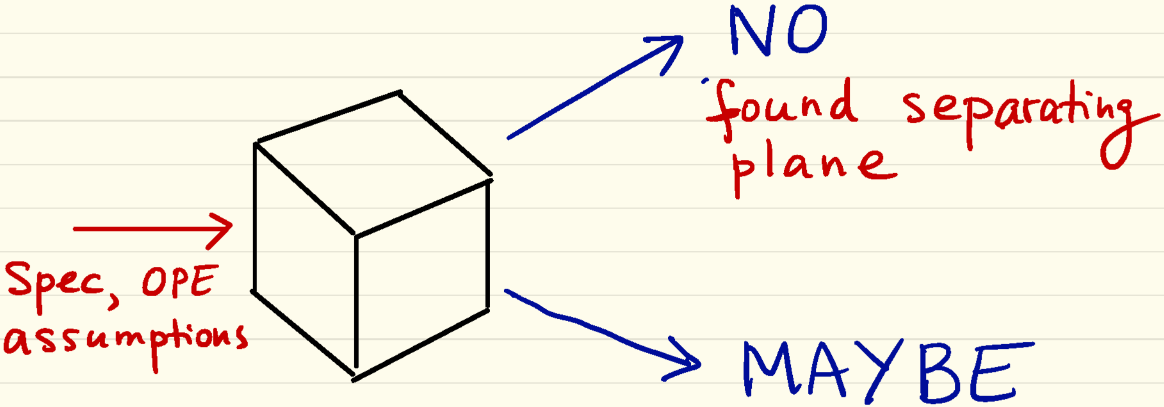
$$\alpha\,(RHS) < 0$$

$$\alpha \circ \mathcal{L}_{\mathcal{O}} \geq 0 \qquad \forall\ \mathcal{O} \in \text{Spectrum}$$

NUMERICAL STRATEGIES

- LOOKING FOR A SEPARATING PLANE is A PROBLEM OF LINEAR / SEMIDEFINITE PROGRAMMING
- EFFICIENT ALGORITHMS EXIST (NEED TO MODIFY SINCE ∞-dim VECTORS $f(u,v)$ & ∞ many CONSTRAINTS $\vartheta_{\Delta,\ell}$)

▶ CAN CONSTRUCT ORACLES



Best current oracle

SDPB [Simmons-Duffin, 1502.02033]

Example:

Is there a unitary 3d CFT with a scalar operator σ, $\Delta_\sigma = \frac{3}{5}$ and OPE $\sigma \times \sigma = \mathbb{1} + \lambda_{\sigma\sigma\varepsilon}\, \varepsilon + \ldots$ where the lowest operator ε has dimension $\Delta_\varepsilon \geq 2$

ORACLE : NO!

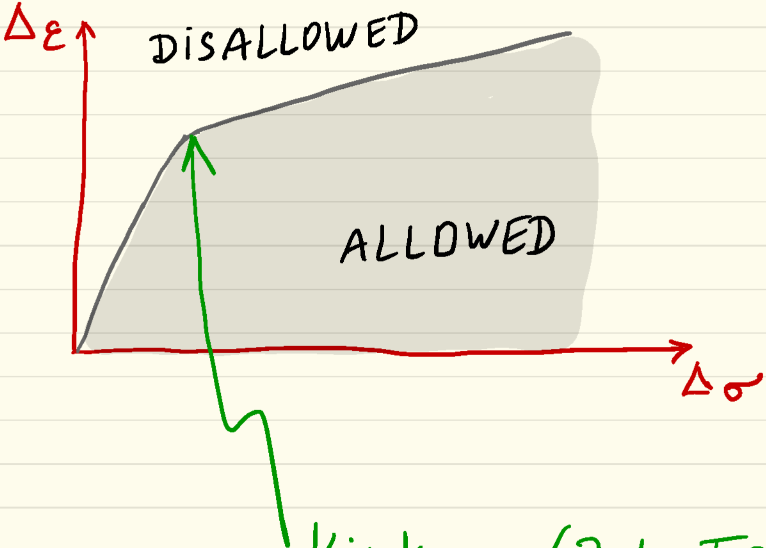
(from studying $\langle \sigma\sigma\sigma\sigma \rangle$)

BOOTSTRAP COMPUTATIONS

- Allowed regions & bounds
- kinks
- islands

BOUND : TRY TO MINIMIZE / MAXIMIZE SOMETHING (Δ or λ)

Example: $\sigma \times \sigma = \mathbb{1} + \lambda_{\sigma\sigma\varepsilon}\, \varepsilon + \ldots$



Kink (3d Ising)

OPERATOR DECOUPLING

ISLANDS

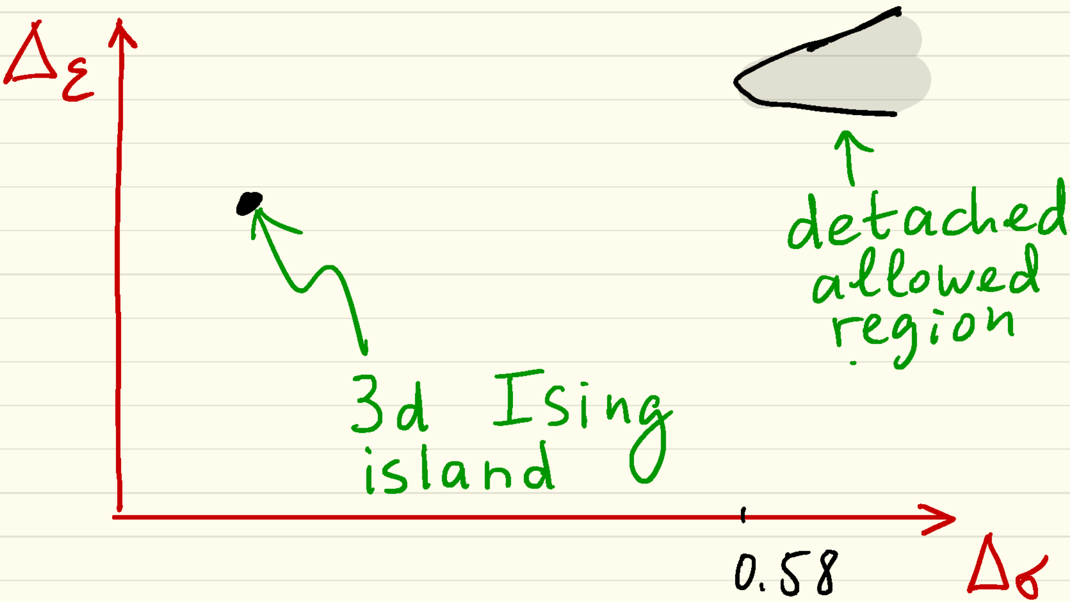
require more assumptions (physically motivated)

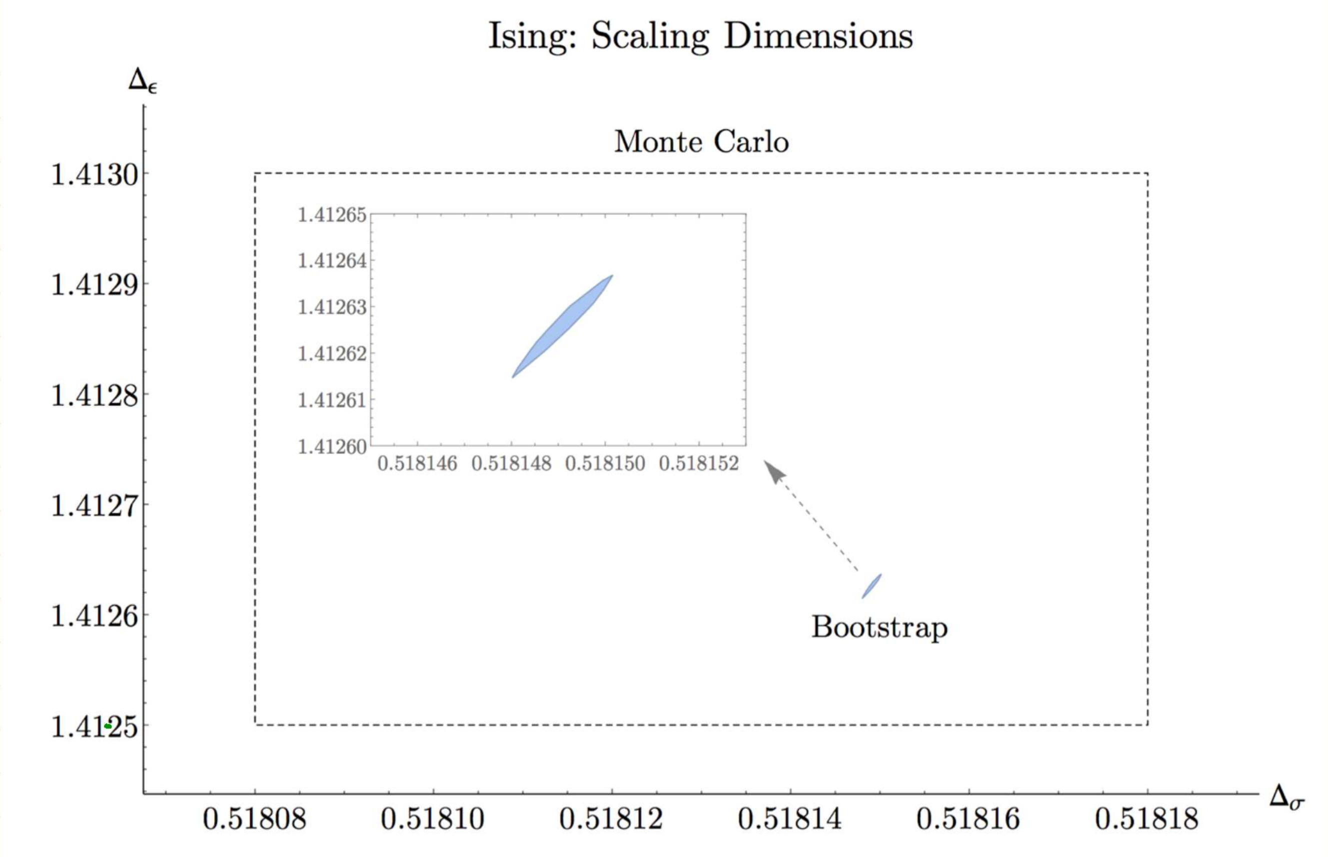
Example

- σ, ε only 2 relevant scalars
- $\mathbb{Z}_2$ symmetry

$$\begin{cases} \sigma \times \sigma = \mathbb{1} + \lambda_{\sigma\sigma\varepsilon}\, \varepsilon + (\Delta \geq 3) \\ \sigma \times \varepsilon = \lambda_{\sigma\sigma\varepsilon}\, \sigma + (\Delta \geq 3) \\ \varepsilon \times \varepsilon = \mathbb{1} + \lambda_{\varepsilon\varepsilon\varepsilon}\, \varepsilon + (\Delta \geq 3) \end{cases}$$

study $\langle \sigma\sigma\sigma\sigma \rangle$, $\langle \sigma\sigma\varepsilon\varepsilon \rangle$, $\langle \varepsilon\varepsilon\varepsilon\varepsilon \rangle$





Kos, Poland, Simmons-Duffin, Vichi '2016

$\mathcal{O}$	$\mathbb{Z}_2$	ℓ	Δ	$f_{\sigma\sigma\mathcal{O}}$	$f_{\epsilon\epsilon\mathcal{O}}$
ϵ	$+$	0	1.412625**(10)**	1.0518537**(41)**	1.532435**(19)**
ϵ'	$+$	0	3.82968(23)	0.053012(55)	1.5360(16)
	$+$	0	6.8956(43)	0.0007338(31)	0.1279(17)
	$+$	0	7.2535(51)	0.000162(12)	0.1874(31)
$T_{\mu\nu}$	$+$	2	3	0.32613776(45)	0.8891471(40)
$T'_{\mu\nu}$	$+$	2	5.50915(44)	0.0105745(42)	0.69023(49)
	$+$	2	7.0758(58)	0.0004773(62)	0.21882(73)
$C_{\mu\nu\rho\sigma}$	$+$	4	5.022665(28)	0.069076(43)	0.24792(20)
	$+$	4	6.42065(64)	0.0019552(12)	$-0.110247(54)$
	$+$	4	7.38568(28)	0.00237745(44)	0.22975(10)
	$+$	6	7.028488(16)	0.0157416(41)	0.066136(36)

$\mathcal{O}$	$\mathbb{Z}_2$	ℓ	Δ	$f_{\sigma\epsilon\mathcal{O}}$	-
σ	$-$	0	0.5181489**(10)**	1.0518537**(41)**	
σ'	$-$	0	5.2906(11)	0.057235(20)	
	$-$	2	4.180305(18)	0.38915941(81)	
	$-$	2	6.9873(53)	0.017413(73)	
	$-$	3	4.63804(88)	0.1385(34)	
	$-$	4	6.112674(19)	0.1077052(16)	
	$-$	5	6.709778(27)	0.04191549(88)	

Simmons-Duffin
1612.08471

~ 100 operators

What are these numbers ?

Exact transcendental critical exponents?

Rong, Su 1807.04434
Atanasov, Hillman, Poland 1807.05702

- Studied $\mathcal{N}=1$ SUSY 3d Ising model:

$$\sigma \times \sigma = \mathbb{1} + \# \varepsilon + \# \varepsilon'$$

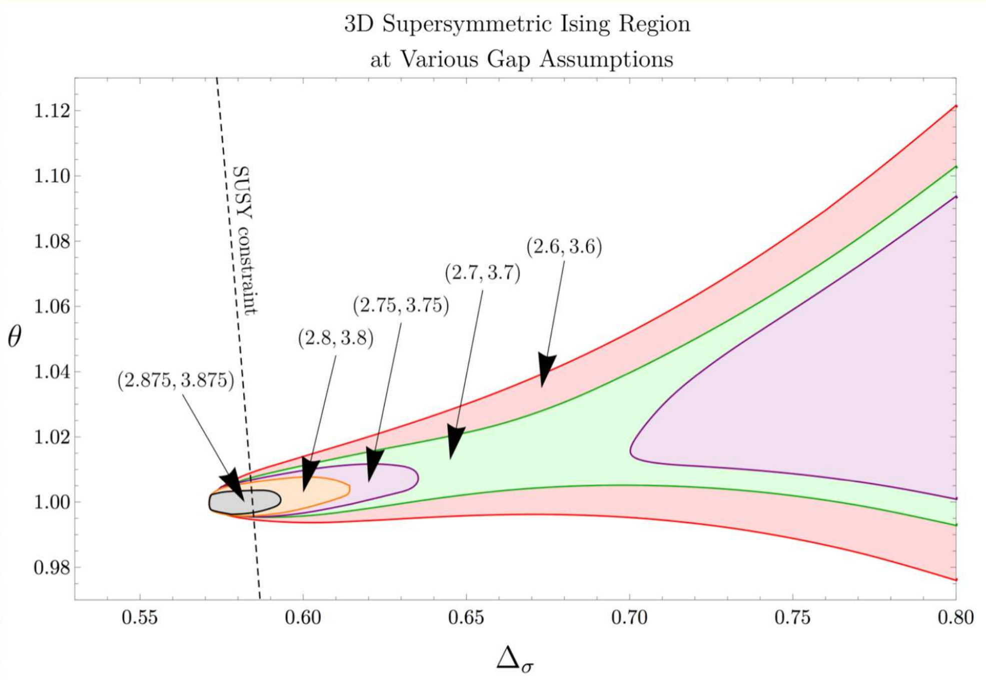
$$\varepsilon \times \varepsilon = \mathbb{1} + \# \varepsilon + \# \varepsilon'$$

$$\sigma \times \varepsilon = \mathbb{1} + \# \sigma + \# \sigma'$$

relevant

$$\Delta_\varepsilon = \Delta_\sigma + 1$$
$$\Delta_{\varepsilon'} = \Delta_{\sigma'} + 1$$

- Found an island:



- Conclude with a **bold conjecture**

$$\frac{\lambda_{\epsilon\epsilon\epsilon}}{\lambda_{\sigma\sigma\epsilon}} = \tan(1)$$

$$\Delta_\sigma = \frac{15 - 2\tan(1) - \sqrt{4\tan(1)^2 + 36\tan(1) + 9}}{18 - 8\tan(1)}$$

$$\approx 0.5844513369 6\ldots$$

CONCLUSIONS

- CONFORMAL BOOTSTRAP WORKS
- WEALTH OF PRECISE NUMERICAL DATA ABOUT STRONGLY COUPLED 3D CFTs

WHAT'S IN IT FOR YOU ?

- CURIOSITY
- TRY TO JUSTIFY BASIC ASSUMPTIONS LEADING TO THE BOOTSTRAP EQNS
- ANALYTIC UNDERSTANDING OF BOOTSTRAP BOUNDS, KINKS, ISLANDS

THAT'S WHERE HELP MOST NEEDED

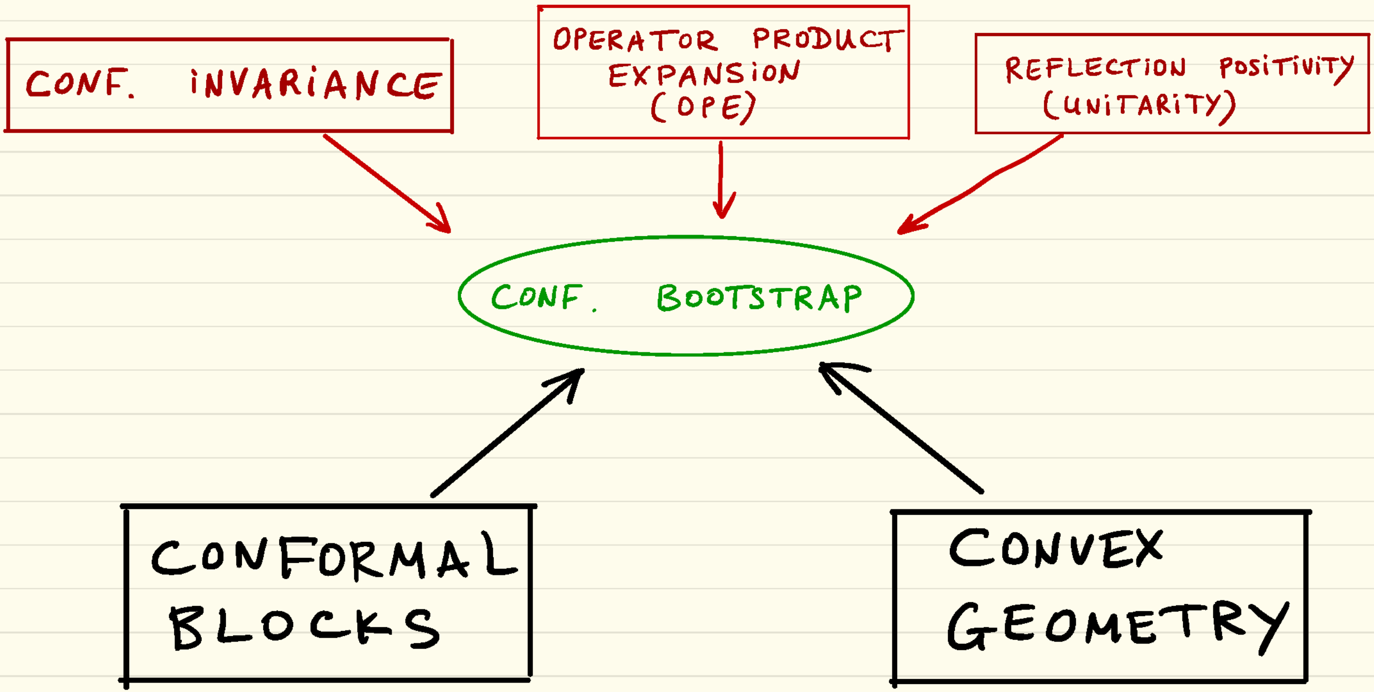
CF. MAZAC 1611.10060
MAZAC, PAULOS 1803.10233

SPHERE PACKING IN D=8

- Cohn, Elkies :
  numerical upper bounds using
  linear programming
  $\Rightarrow$ magic function ?

- Viazovska '2016 :
  constructed magic function
  using modular form.

Will this happen for bootstrap ?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