The triple reduced product and Higgs bundles

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Introduction

Outline:

1. The triple reduced product space: Motivation
2. Higgs bundles
3. The Higgs field
4. Lax form
5. Hamiltonian flow
6. Hamiltonian flow for circle action

I. The TRP Space: Motivation

- Let $G = SU(3)$. We are considering the symplectic quotient of the product of three coadjoint orbits of $SU(3)$

$$M = \mathcal{O}_\lambda \times \mathcal{O}_\mu \times \mathcal{O}_\nu // G$$

  where $\lambda, \mu, \nu \in \mathbf{t}$ are in the Lie algebra of the maximal torus of $SU(3)$ (in other words they are diagonal matrices with purely imaginary entries).

- The moment map for each (co)adjoint orbit is the inclusion map into the Lie algebra $\mathbf{g}$.

- So if $X, Y, Z \in \mathcal{O}_\lambda \times \mathcal{O}_\mu \times \mathcal{O}_\nu$, the moment map is $\phi(X, Y, Z) = X + Y + Z$.

- This space has dimension 2 (because the dimension of each of the orbits is 6 and the moment map condition reduces the dimension by 8 , while the quotient by the group action reduces by a further 8: $18 - 8 - 8 = 2$)

- The triple reduced product space may be identified with a polygon space, a space of triangles in $\mathfrak{su}(3)$ with vertices in specific coadjoint orbits.
- These spaces are a prototype for flat connections on the three-punctured sphere, with the holonomy around each puncture constrained to lie in a prescribed conjugacy class. (See LJ,c *Math. Ann.* 1994.)

- The orbit method (Kirillov) has many applications in geometry.
- A tuple of matrices may be identified with a Higgs field.
- In the paper "The triple reduced product and Hamiltonian flows" (L. Jeffrey, S. Rayan, G. Seal, P. Selick, J. Weitsman, in XXXV WGMP Proceedings), the main objective was to identify a Hamiltonian function which was the moment map for a circle action. We were able to do this only indirectly, by choosing an auxiliary function which maps the triple reduced product onto the unit interval, and defining the moment map indirectly as a definite integral involving the auxiliary function.
- Identifying the triple reduced product as a subset of the space of Higgs bundles gives us another method. A tuple of matrices may be identified with a Higgs field. Polygon spaces are known to live within parabolic Hitchin systems. We show that Hamiltonian circle actions arise naturally through Hitchin systems; see for instance Adams-Harnad-Hurtubise (CMP 1990), Biswas-Ramanan (JLMS 1994), E. Markman (Comp. Math. 1994).
- We recall also the embedding of the triple reduced product in a loop algebra (Adler-van Moerbeke, Reyman-Semenov-Tian-Shansky, Mischenko-Fomenko).
- Symplectic volume of triple reduced product is known (Suzuki-Takakura '08; LJ- Jia Ji, arXiv:1804.06474)

- Assuming that 0 is a regular value of the moment map, the triple reduced product is homeomorphic to S^2.

2. Higgs bundles

- We may identify the triple reduced product with a compact space of Higgs bundles over $\mathbb{C}P^1 \setminus \{0, 1, -1\}$ where the residues of the Higgs fields are constrained to live in the fixed coadjoint orbits $\mathcal{O}_\lambda, \mathcal{O}_\mu, \mathcal{O}_\nu$).
- A Higgs bundle is a pair (P, Φ) where P is a holomorphic principal $SU(3)$ bundle over $\mathbb{C}P^1$ and Φ is a meromorphic map from $\mathbb{C}P^1$ to $ad(P) \otimes K(D)$, where $K \cong \mathcal{O}(-2)$ is the canonical line bundle. Here D is the divisor $0 + 1 + (-1)$ (consisting of the three marked points).
- For each $z \in \mathbb{C}P^1$, $z \neq 0, \pm 1$, $\Phi(z)$ is trace-free and anti-Hermitian.

3. The spectral curve

- We restrict to the set of bundles P where P is topologically trivial.

  So we can write

$$\Phi(z) = (\frac{X}{z} + \frac{Y}{z-1} + \frac{Z}{z+1})dz$$

  or ignore the dz and write

$$\Phi(z) = \frac{X}{z} + \frac{Y}{z-1} + \frac{Z}{z+1}$$

- Let

$$L(z) = z(z-1)(z+1)\Phi(z) = (Y-Z)z + (Y+Z).$$

  Then

$$\rho(z,\eta) = \det\left(L(z) - \eta I\right)$$

- The spectral curve is obtained by setting

$$\rho(z,\eta) = 0$$

- The *Hitchin map* sends Φ to its characteristic polynomial :

$$\Phi \mapsto \det(\Phi - \eta I).$$

- The Hitchin map sends Φ to an affine space whose dimension is half the dimension of M. The fiber of the Hitchin map is the space of bundles of a fixed degree.

- It turns out that the spectral curve is a Riemann surface of genus 1.

- The spectral curve is invariant under the involution

$$(z, \eta) \mapsto (\bar{z}, -\bar{\eta})$$

- If we impose the restriction that the residues X, Y, Z of Φ lie in the coadjoint orbits $\mathcal{O}_\lambda, \mathcal{O}_\mu, \mathcal{O}_\nu$, the space P is identified with the triple reduced product.

- The constraint that $X + Y + Z = 0$ comes from the constraint that the trace of Φ is zero. It is simply the condition that there are no poles at infinity.

4. Lax form

- A function H gives a Hamiltonian flow along the triple reduced product which can be written in Lax form.
- The Hamiltonian flow within P is

$$\frac{d\Phi}{dt} = [dH, \Phi]$$

- This gives

$$\frac{dLs}{dt} = i\left[(Y-Z)^2 z + (Y-Z)(Y+Z) + (Y+Z)(Y-Z), (Y-Z)z + (Y+Z)\right]$$

- This leads to

$$\frac{d(Y-Z)}{dt} = 0$$

$$\frac{d(Y+Z)}{dt} = i\left[(Y+Z)(Y-Z), Y+Z\right]$$

or

$$\frac{dY}{dt} = i[Y, YZ + ZY + Z^2]$$

- There is a similar equation for $\frac{dZ}{dt}$. We conclude that Y, Z and $Y+Z$ evolve by conjugation, so $\Phi(z)$ and $L(z)$ evolve by conjugation.

5. Hamiltonian Flow

- The Higgs field may be described by

$$L(z) = Az + S$$

where

$$A = Y - Z, \quad S = Y + Z$$

in terms of the elements $Y, Z \in \mathfrak{g}$. So A is a diagonal matrix with eigenvalues $\alpha_1, \alpha_2, \alpha_3$. Define $s_{i,j}$ as the entries of the matrix S.

- Let $\tilde{L}(z, \eta)$ be the matrix of cofactors of $L(z) - \eta I$:

$$(L(z) - \eta)\tilde{L}(z, \eta) = \rho(z, \eta)I$$

- Here

$$\rho(z, \eta) = \det\left(z(z^2 - 1)\Phi(z)\right) = iHz(z^2 - 1) + Q_0(z) + Q_1(z)\eta - \eta^3,$$

where the $Q_j(z)$ are quadratic functions of z. The function iH can be taken to be $\det(Y - Z)$, or $-i\text{Trace}(Y - Z)^3/3$. It is the only coefficient of the Hitchin map which is not constant on the triple reduced product.

- Here L is linear in z.

- Then we obtain

$$\tilde{L}_{21}(z,\eta) = -s_{2,1}(\alpha_3 z - \eta + s_{3,3}) + s_{3,1}s_{2,3}$$

$$\tilde{L}_{3,1}(z,\eta) = -s_{3,1}(\alpha_2 z - \eta + s_{2,2}) + s_{2,1}s_{3,2}$$

- Setting these two equations equal to zero we get a unique solution set z_0, η_0 leading to unique solutions z_0, ζ_0 with $z_0((z_0)^2 - 1)\zeta_0 = \eta_0$.
- It is also true that $\tilde{L}_{1,1}(z_0, \eta_0) = 0$
- It was shown by M. Adams, J. Harnad and J. Hurtubise (Lett. Math. Phys. 1997) that z_0 and ζ_0 are Darboux coordinates for this system. See also papers of the same authors in Commun. Math. Phys. 1990, 1993.

6. Constructing the Hamiltonian flow

- Let

$$G(z,H) := \int^{z_0} \zeta(z,H)dz.$$

- Then

$$\frac{\partial G}{\partial z_0} = \zeta_0.$$

- If we define

$$t(z_0) = \frac{\partial G}{\partial H} = \int_0^{z_0} \frac{\partial\rho/\partial H}{z(z^2-1)\partial\rho/\partial\eta}dz = \int_0^{z_0} \frac{dz}{Q_1(z)-3\eta^2}.$$

- This follows because (by the chain rule)

$$\frac{\partial \rho}{\partial H} = \frac{\partial \rho}{\partial \eta}\frac{\partial \eta}{\partial H}.$$

If we flow around a closed cycle γ , we find that the period is

$$T(H) = \int_\gamma \frac{dz}{Q_1(z)-3\eta^2}.$$

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- There is a function $F(H)$ whose Hamiltonian flow generates the S^1 action (because H is constant under the S^1 action).
- The Hamiltonian vector field is

$$X_V = \frac{dF}{dH} X_H$$

- So the period of F is the period of H divided by dF/dH.
- It follows (since the period of F is 1) that the period of H is

$$T(H) = \frac{dF}{dH}.$$

- Examining the equation for G and looking at $\partial G/\partial z_0 = \zeta_0$, we have

$$F(H) = \int_\gamma \eta(a, H) dz.$$

- Since z and ζ are Darboux coordinates, we have found action-angle variables for our system.

7. Hamiltonian for circle action

- Starting from

$$\alpha_3 z_0 - \eta_0 = \frac{s_{3,1}s_{2,3}}{s_{2,1}} - s_{3,3}$$

$$\alpha_1 z_0 - \eta_0 = \frac{s_{2,1}s_{3,2}}{s_{3,1}} - s_{2,2}$$

  we subtract to get

$$z_0 = \frac{1}{\alpha_3 - \alpha_2}\left(\frac{s_{2,1}s_{3,2}}{s_{3,1}} - \frac{s_{3,1}s_{2,3}}{s_{2,1}} - s_{2,2} + s_{3,3}\right)$$

- The function $F(H)$ takes values in an interval.

The symplectic volume of the reduced product

Joint work with Jia Ji (University of Toronto)

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Outline:

1. Background
2. $SU(3)$, $N = 3$
3. Generalizations
4. Suzuki-Takakura

References

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1. Background

Let $\xi_1, \ldots, \xi_N \in \mathbf{t}$. **Assumption 1:** All of $\mathcal{O}(\xi_1), \cdots, \mathcal{O}(\xi_N)$ are diffeomorphic to the homogeneous space G/T. This assumption is equivalent to the assumption that all of the stabilizer groups $\mathrm{Stab_G(\xi_1), \cdots, Stab_G(\xi_N)}$ are conjugate to the chosen maximal torus T. If all of $\xi_1, \cdots, \xi_N$ are contained in $\mathfrak{t}^* \subseteq \mathfrak{g}^*$, then this assumption is saying that

$$\mathrm{Stab_G(\xi_1) = \cdots = Stab_G(\xi_N) = T.}$$

Remark: Since every coadjoint orbit $\mathcal{O}(\xi)$ can be written as $\mathcal{O}(\xi')$ for some $\xi' \in \mathfrak{t}^* \subseteq \mathfrak{g}^*$, we can always assume that $\underline{\xi} = (\xi_1, \cdots, \xi_N)$ satisfies that $\xi_j \in \mathfrak{t}^* \subseteq \mathfrak{g}^*$ for all j.

The Cartesian product $\mathcal{M}(\underline{\xi}) = \mathcal{O}(\xi_1) \times \cdots \times \mathcal{O}(\xi_N)$ carries a natural symplectic structure $\omega_{\underline{\xi}}$ defined by:

$$\omega_{\underline{\xi}} := \pi_1^* \omega_{\mathcal{O}(\xi_1)} + \cdots + \pi_N^* \omega_{\mathcal{O}(\xi_N)} \tag{1}$$

where $\pi_j : \mathcal{O}(\xi_1) \times \cdots \times \mathcal{O}(\xi_N) \to \mathcal{O}(\xi_j)$ is the projection onto the j-th component.

Let G act on $\mathcal{M}(\underline{\xi}) = \mathcal{O}(\xi_1) \times \cdots \times \mathcal{O}(\xi_N)$ by the diagonal action Δ:

$$\Delta(g)(\eta_1, \cdots, \eta_N) := (K(g)(\eta_1), \cdots, K(g)(\eta_N)) \tag{2}$$

for all $g \in G$, $\eta_j \in \mathcal{O}(\xi_j)$. Here $K(g)$ denotes the (co)adjoint action of g.

The symplectic form $\omega_{\underline{\xi}}$ is clearly G-invariant, and we also have the following.

Proposition: The diagonal action Δ of G on $(\mathcal{M}(\underline{\xi}), \omega_{\underline{\xi}})$ is a Hamiltonian G-action with the moment map $\mu_{\underline{\xi}} : \mathcal{M}(\underline{\xi}) \to \mathfrak{g}^*$ being:

$$\mu_{\underline{\xi}}(\underline{\eta}) = \sum_{j=1}^{N} \eta_j \tag{3}$$

for all $\underline{\eta} := (\eta_1, \cdots, \eta_N) \in \mathcal{M}(\underline{\xi})$.

We assume that:

Assumption 2: $0 \in \mathfrak{g}^*$ is a regular value for $\mu_{\underline{\xi}} : \mathcal{M}(\underline{\xi}) \to \mathfrak{g}^*$ and $\mu_{\underline{\xi}}^{-1}(0) \neq \emptyset$.

Remark: By Sard's theorem, the set where the previous two assumptions hold is nonempty and has nonempty interior in $\mathfrak{t}^* \times \cdots \times \mathfrak{t}^*$.

Then, the level set $\mathcal{M}_0(\underline{\xi}) := \mu_{\underline{\xi}}^{-1}(0)$ is a closed, thus compact, submanifold of $\mathcal{M}(\underline{\xi})$ and the diagonal action Δ of G restricts to an action on $\mathcal{M}_0(\underline{\xi})$. Therefore, we can form the quotient space with respect to this action of G on $\mathcal{M}_0(\underline{\xi})$:

$$\mathcal{M}(\underline{\xi}) := \mathcal{M}_0(\underline{\xi})/G. \tag{4}$$

Sometimes, the above quotient space is also denoted by $M//G$. Note that this quotient space is compact.

If the G-action on $\mathcal{M}_0(\underline{\xi})$ is free and proper (in our situation, properness is automatically satisfied), then the quotient space $\mathcal{M}(\underline{\xi}) = \mathcal{M}_0(\underline{\xi})/G$ is a smooth manifold. However, in our situation, the G-action on $\mathcal{M}_0(\underline{\xi})$ is in general not free. Hence, in general the quotient space is only an orbifold. To avoid this complication, we will assume:

Assumption 3: The quotient space $\mathcal{M}(\underline{\xi}) = \mathcal{M}_0(\underline{\xi})/G$ is a smooth compact manifold.

Remark: The above assumption will put further restrictions on which $\underline{\xi} \in \mathfrak{t}^* \times \cdots \times \mathfrak{t}^*$ we can choose as initial data. Thus we only choose initial data from the following set in this talk:

$$\mathcal{A}' := \left\{ \underline{\xi} \in \overbrace{\mathfrak{t}^* \times \cdots \times \mathfrak{t}^*}^{N} \ : \ \text{previous 3 assumptions hold} \right\} \tag{5}$$

Suzuki and Takakura also made this assumption in their paper [ST] (in Section 2.3). It seems reasonable to us to assume that even after Assumption 3 is imposed, the initial data set $\mathcal{A}'$ is still nonempty and still has nonempty interior in $\mathfrak{t}^* \times \cdots \times \mathfrak{t}^*$. Notice that since the elements in the center of G always act trivially on $\mathcal{M}(\underline{\xi})$ and $\mathcal{M}_0(\underline{\xi})$, Assumption 3 is valid if $PG = G/Z(G)$ acts freely on $\mathcal{M}_0(\underline{\xi})$. This happens for $G = SU(n)$ if all the coadjoint orbits $\mathcal{O}(\xi_i)$ are generic.

Then, we have the following well known theorem:

Theorem:[Marsden-Weinstein] The smooth compact manifold $\mathcal{M}(\underline{\xi}) = \mathcal{M}_0(\underline{\xi})/G$ carries a unique symplectic structure $\omega(\underline{\xi})$ such that

$$i^*\omega_{\underline{\xi}} = \pi^*\omega(\underline{\xi}) \tag{6}$$

where $i : \mathcal{M}_0(\underline{\xi}) \hookrightarrow \mathcal{M}(\underline{\xi})$ is the inclusion map and $\pi : \mathcal{M}_0(\underline{\xi}) \to \mathcal{M}(\underline{\xi})$ is the associated projection map.

2. The case $G = SU(3)$, $N = 3$

In this section, we study 3-fold reduced products, or triple reduced products for $G = SU(3)$. See [TRP1], [TRP2] for recent studies about these objects. Our focus is on the symplectic volume of triple reduced products.

The Setup for the case $G = SU(3)$

The setup here is due to Suzuki-Takakura [ST].

- Let $G = SU(3)$ and let T be its standard maximal torus, i.e., T consists of diagonal matrices in $SU(3)$.
- In this case, we know that the corresponding Weyl group W is isomorphic to the permutation group $\mathfrak{S}_3$.
- The Weyl group W acts on $\mathfrak{t}^* \cong \mathfrak{t}$ by permutations of diagonal entries.
- The elements

$$H_1 := 2\pi i \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H_2 := 2\pi i \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

  in $\mathfrak{t}$ are generators of the integral lattice $\exp^{-1}(I) \subset \mathfrak{t}$. The elements H_1, H_2 form a basis of $\mathfrak{t}$.

- Let ω_1, ω_2 be the basis of $\mathfrak{t}^*$ dual to H_1, H_2, i.e., $\omega_i(H_j) = \delta_{ij}$. Under the identification $\mathfrak{t}^* \cong \mathfrak{t}$, ω_1, ω_2 correspond to the elements

$$\Omega_1 := \frac{2\pi i}{3}\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \Omega_2 := \frac{2\pi i}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

  in $\mathfrak{t}$, respectively.

- Let $\mathfrak{t}^*_+ := \mathbb{R}_{\geq 0}\omega_1 + \mathbb{R}_{\geq 0}\omega_2$ and $\Lambda_+ := \mathbb{Z}_{\geq 0}\omega_1 + \mathbb{Z}_{\geq 0}\omega_2$. So $\mathfrak{t}^*_+$ is a positive Weyl chamber and Λ_+ is the associated set of dominant integral weights. Any element ξ of $\mathfrak{t}^*_+$ or Λ_+ can be written as

$$\xi = (\ell - m)\omega_1 + m\omega_2, \quad \ell \geq m \geq 0. \tag{7}$$

- Under the identification $\mathfrak{t}^* \cong \mathfrak{t}$, ξ corresponds to the element

$$X = (\ell - m)\Omega_1 + m\Omega_2. \tag{8}$$

- Every coadjoint orbit can be written as $\mathcal{O}_\xi$ for some $\xi \in FWC$, and in this case, $\mathcal{O}_\xi \cap \mathfrak{t}^*$ is the W-orbit through ξ, and $\mathcal{O}_\xi \cap FWC = \{\xi\}$.

- If $\xi = (\ell - m)\omega_1 + m\omega_2 \in FWC$ with $\ell > m > 0$, then $\mathrm{Stab}_{\mathrm{G}}(\xi) = \mathrm{T}$ and $\mathcal{O}_\xi$

is diffeomorphic to the homogeneous space G/T.

- Let $\xi_1, \xi_2, \xi_3 \in FWC$ so that $\xi_i = (\ell_i - m_i)\omega_1 + m_i\omega_2$ with $\ell_i > m_i > 0$. Let $\underline{\xi} := (\xi_1, \xi_2, \xi_3)$.
- Then $\underline{\xi}$ determines a triple reduced product $(\mathcal{M}(\underline{\xi}), \omega(\underline{\xi}))$.

Symplectic Volume of a Triple Reduced Product

- By nonabelian localization [JK95], $\int_{\mathcal{M}} e^{i\omega}$ can be expressed as a finite sum of contributions indexed by the fixed point set $\mathcal{M}^T$ of $\mathcal{M}$ under the action of the maximal torus T:

$$\mathcal{M}^T = \{(w_1 \cdot \xi_1, w_2 \cdot \xi_2, w_3 \cdot \xi_3) \,:\, w_1, w_2, w_3 \in W\}\,. \tag{9}$$

- More precisely, we have

$$\int_{\mathcal{M}} e^{i\omega} = \sum_{\underline{w} \in W^3} \int_{X \in \mathbf{t}} \varpi^2(X) \frac{e^{i\langle \mu(\underline{w} \cdot \underline{\xi}), X \rangle}}{\mathrm{e}_{\underline{w} \cdot \underline{\xi}}(X)}\, dX, \tag{10}$$

- Here
- $\underline{w} = (w_1, w_2, w_3) \in W^3, \underline{\xi} = (\xi_1, \xi_2, \xi_3)$

$$\underline{w} \cdot \underline{\xi} := (w_1 \cdot \xi_1, w_2 \cdot \xi_2, w_3 \cdot \xi_3), \tag{11}$$

- $\varpi(X) = \prod_\alpha \langle \alpha, X \rangle$ with α running over all positive roots of $G = SU(3)$
- $\mathrm{e}_F(X)$ is the equivariant Euler class of the normal bundle to the fixed point F. In this case,

$$\mathrm{e}_{\underline{w} \cdot \underline{\xi}}(X) = sgn(\underline{w}) \varpi^3(X), \tag{12}$$

where $\mathrm{s}gn(\underline{w}) := \mathrm{s}gn(w_1)\mathrm{s}gn(w_2)\mathrm{s}gn(w_3)$.

- This is the Fourier transform of the Duistermaat-Heckman oscillatory integral evaluated at 0.

- The DH oscillatory integral decomposes as a sum of finitely many terms. None of these terms separately admits a Fourier transform, but it is possible to define a Fourier transform of each term provided one picks polarizations consistently at each term (Guillemin-Lerman-Sternberg, op. cit., 1983).
- In the special case when $\mathbf{t} = \mathbf{R}$, a choice of a polarization is a choice to replace $\mathbf{R}$ by $\mathbf{R} + \mathbf{i}\epsilon$ where the choice of polarization is the choice of sign of ϵ.

Theorem:

$$\int_{\mathcal{M}} e^{i\omega} = \sum_{\underline{w}\in W^3} sgn(\underline{w}) \int_{X\in\mathbf{t}} \frac{e^{i\left\langle \mu(\underline{w}\cdot\underline{\xi}),X\right\rangle}}{\varpi(X)}\, dX. \tag{13}$$

The symplectic volume of the reduced space $\mu_{\eta,T}^{-1}(0)/T$ of the Hamiltonian system $(\mathcal{O}_\eta, \omega_\eta, T, \mu_{\eta,T})$, where $\mu_{\eta,T} : \mathcal{O}_\eta \hookrightarrow \mathbf{t}^* \subset \mathbf{g}^*$ is the moment map associated to the Hamiltonian group action (in this case, the coadjoint action) on $\mathcal{O}_\eta$ by the standard maximal torus T, is expressed by the following formula, known from [GLS] and [JK95] (using Atiyah-Bott-Berline-Vergne localization).

- **Theorem:**

$$\mathrm{S}Vol(\mu_{\eta,T}^{-1}(0)/T) = \frac{1}{2\pi i}\sum_{w\in W}\mathrm{s}gn(w)\int_{X\in\mathbf{t}}\frac{e^{i\langle w\cdot\eta,X\rangle}}{\varpi(X)}\,dX. \tag{14}$$

- Let

$$f(\eta) := 2\pi i\,\mathrm{Vol}(\mu_{\eta,T}^{-1}(0)/T) = \sum_{w\in W}\mathrm{s}gn(w)\int_{X\in\mathbf{t}}\frac{e^{i\langle w\cdot\eta,X\rangle}}{\varpi(X)}\,dX. \tag{15}$$

- Then, by writing $w_2 = w_1w_1^{-1}w_2,\ w_3 = w_1w_1^{-1}w_3$ and letting $w_2' = w_1^{-1}w_2,\ w_3' = w_1^{-1}w_3$, we obtain

$$\int_{\mathcal{M}} e^{i\omega} = \sum_{w_2'\in W}\sum_{w_3'\in W}\mathrm{s}gn(w_2')\mathrm{s}gn(w_3')f(\xi_1 + w_2'\cdot\xi_2 + w_3'\cdot\xi_3). \tag{16}$$

- On the other hand, it is known from [JK95] (from Atiyah-Bott-Berline-Vergne localization formula and nonabelian localization) that

Theorem:

$$\mathrm{Vol}(\mu^{-1}_{\eta,T}(0)/T) = \sum_{w \in W} \mathrm{sgn}(w) H_{\underline{\beta}}(w \cdot \eta) \tag{17}$$

- Here $\underline{\beta} = (\beta_1, \beta_2, \beta_3)$ and $\beta_1, \beta_2, \beta_3$ are the positive roots of $SU(3)$, and

$$H_{\underline{\beta}}(\xi) := \mathrm{vol}_a \left\{ (s_1, s_2, s_3) \in \mathbb{R}^3_{\geq 0} \ : \ \sum_{i=1}^{3} s_i \beta_i = \xi \right\} \tag{18}$$

- Here, vol_a here denotes the standard a-dimensional Euclidean volume multiplied by a normalization constant, and

$$a = r - \dim T \tag{19}$$

- Here r is the number of positive roots of $SU(3)$. Notice that in this case $a = 1$.

- Therefore $\int_{\mathcal{M}} e^{i\omega}$ can also be expressed as

$$2\pi i \sum_{w_2' \in W} \sum_{w_3' \in W} \mathrm{sgn}(w_2') \mathrm{sgn}(w_3') \sum_{w_1 \in W} \mathrm{sgn}(w_1) H_{\underline{\beta}}(w_1 \cdot (\xi_1 + w_2' \cdot \xi_2 + w_3' \cdot \xi_3)). \tag{20}$$

- By letting $w_2' = w_1^{-1} w_2,\ w_3' = w_1^{-1} w_3$, we then obtain

$$\int_{\mathcal{M}} e^{i\omega} = 2\pi i \sum_{\underline{w} \in W^3} sgn(\underline{w}) H_{\underline{\beta}}(\mu(\underline{w} \cdot \underline{\xi})). \tag{21}$$

So we obtain the volume formula for triple reduced products for $G = SU(3)$:

Theorem:

$$\mathrm{S}Vol(\mathcal{M}(\underline{\xi})) = \sum_{\underline{w} \in W^3} \mathrm{s}gn(\underline{w}) H_{\underline{\beta}}(\mu(\underline{w} \cdot \underline{\xi})). \tag{22}$$

Here, $H_{\underline{\beta}} : \mathfrak{t}^* \to \mathbf{R}$ is called the Duistermaat-Heckman function. For a general semisimple compact connected Lie group G, it can be defined as follows:

- **Definition:**

$$H_{\underline{\beta}}(\xi) = \mathrm{vol}_a \left\{ (s_1, \cdots, s_r) \ : \ s_i \geq 0, \quad \sum_{i=1}^{r} s_i \beta_i = \xi \right\} \tag{23}$$

- $\underline{\beta} = (\beta_1, \cdots, \beta_r)$ and $\beta_1, \cdots, \beta_r \in \mathfrak{t}^*$ are all the positive roots of G and $a = r - \dim T$.

- For $G = SU(3)$, there are two simple roots β_1, β_2 (with $< \beta_1, \beta_2 >= 2\pi/3$) and one additional positive root $\beta_3 = \beta_1 + \beta_2$. Expressing ξ as (ξ_1, ξ_2) where $\xi_j =< \beta_j, \xi >$, we have

$$H_{\underline{\beta}}(\xi) = \xi_1$$

  if $\xi_2 > \xi_1$ and

$$H_{\underline{\beta}}(\xi) = \xi_2$$

  if $\xi_1 > \xi_2$. Notice that the two definitions agree when $\xi_1 = \xi_2$. The function is continuous along that line, but its first derivatives are not continuous there.

- **Remark:** It is clear from the above definition that $H_{\underline{\beta}}$ is supported in the cone

$$C_{\underline{\beta}} := \left\{ \sum_{i=1}^{r} s_i \beta_i \; : \; s_i \geq 0 \right\} \subseteq \mathfrak{t}^*. \tag{24}$$

- In the case $G = SU(3)$, we have $r = 3$ and
- $$\beta_1 = H_1,\ \beta_2 = H_2,\ \beta_3 = H_1 + H_2.$$
- If $\xi = (\ell - m)\Omega_1 + m\Omega_2 = (\ell - m) \cdot (2H_1 + H_2)/3 + m \cdot (H_1 + 2H_2)/3$, then we obtain:
  $$H_{\underline{\beta}}(\xi) = \kappa \cdot \max\left\{ \min\left\{ \frac{2}{3}\ell - \frac{1}{3}m,\ \frac{1}{3}\ell + \frac{1}{3}m \right\},\ 0 \right\} \tag{25}$$
  where κ is a normalization constant.
- We fix the basis $\{\Omega_1, \Omega_2 - \Omega_1\}$ for $\mathfrak{t}$. Then, each $\xi_i = (\ell_i - m_i)\Omega_1 + m_i\Omega_2 \in \mathfrak{t}$ has (ℓ_i, m_i) as its coordinates in this basis. Hence, $\underline{\xi} = (\xi_1, \xi_2, \xi_3)$ can be represented in this basis by the vector
  $$(\ell_1, \ell_2, \ell_3, m_1, m_2, m_3) \in \mathbb{R}^6. \tag{26}$$

Hence, the symplectic volume of a triple reduced product for $G = SU(3)$ can be computed explicitly by the following formula:

- **Theorem:**

$$\begin{aligned} &\mathrm{SV}ol(l_1, l_2, l_3, m_1, m_2, m_3) = \\ &\kappa \sum_{i,j,k=0}^{5} (-1)^{i+j+k} \max\left\{ \min\left\{ (\frac{2}{3}\pi_1 - \frac{1}{3}\pi_2)(P_{ijk}), \right.\right. \\ &\qquad \left.\left. (\frac{1}{3}\pi_1 + \frac{1}{3}\pi_2)(P_{ijk}) \right\},\ 0 \right\} \end{aligned} \tag{27}$$

- Here,

$$P_{ijk}(l_1, l_2, l_3, m_1, m_2, m_3) = v_i \cdot \begin{pmatrix} l_1 \\ m_1 \end{pmatrix} + v_j \cdot \begin{pmatrix} l_2 \\ m_2 \end{pmatrix} + v_k \cdot \begin{pmatrix} l_3 \\ m_3 \end{pmatrix} \tag{28}$$

  and $\pi_1, \pi_2 : \mathbb{R}^2 \to \mathbb{R}$ are the standard projections to the first and second coordinates, respectively.

3. Generalizations

Symplectic volume of triple reduced products for general semisimple compact connected Lie group G

- Our method applies to any semisimple compact connected Lie group G. Therefore the above theorems still hold in this more general situation.
- The set of positive roots is now different and the Duistermaat-Heckman function $H_{\underline{\beta}}$ should be replaced by the general one above.

- **Symplectic volume of N-fold reduced products for general semisimple compact connected Lie group G**
- We can also generalize our results from the triple reduced product (symplectic quotient of product of three orbits) to the N-fold reduced product (symplectic quotient of product of N orbits). The formulas are similar, although we no longer get a piecewise linear function (the formulas are piecewise polynomial).
- **Theorem:**

$$\int_{\mathcal{M}} e^{i\omega} = \sum_{\underline{w} \in W^N} \mathrm{s}gn(\underline{w}) \int_{X \in \mathbf{t}} \frac{e^{i \langle \mu(\underline{w} \cdot \underline{\xi}), X \rangle}}{\varpi^{N-2}(X)} \, dX \tag{29}$$

- where $\underline{\xi} = (\xi_1, \cdots, \xi_N), \underline{w} = (w_1, \cdots, w_N) \in W^N$ and

$$\varpi(X) = \prod_{\alpha} \langle \alpha, X \rangle \tag{30}$$

where α runs over all the positive roots of G.

- **Proof:** In this case, the equivariant Euler class is

$$\mathrm{e}_{\underline{w}\cdot\underline{\xi}}(X) = (\mathrm{s}gn(\underline{w}))^N \varpi^N(X). \tag{31}$$

In addition, the symplectic volume of $\mathcal{M}$ can be computed by a similar formula involving Duistermaat-Heckman functions:

- **Theorem:**

$$\mathrm{S}Vol(\mathcal{M}) = \sum_{\underline{w}\in W^N} \mathrm{s}gn(\underline{w}) H_{(N-2)\cdot\underline{\beta}}(\mu(\underline{w}\cdot\underline{\xi})) \tag{32}$$

Here, $\underline{\beta} = (\beta_1, \cdots, \beta_r)$ with $\beta_1, \cdots, \beta_r$ being all the positive roots of G and the Duistermaat-Heckman function $H_{(N-2)\cdot\underline{\beta}}$ is defined as follows:

-

$$H_{(N-2)\cdot\underline{\beta}}(\xi) := \mathrm{vol}_a \left\{ (s_1^{(1)}, \cdots, s_r^{(1)}, \cdots, s_1^{(N-2)}, \cdots, s_r^{(N-2)}) \ : \ s_i^{(j)} \geq 0 \text{ for all } i \text{ and } j \text{ and } \sum_{j=1}^{N-2}\sum_{i=1}^{r} s_i^{(j)}\beta_i = \xi \right\} \tag{33}$$

where r is the number of positive roots of G and $a = (N-2)\cdot r - \dim T$.

- **Remark:** Notice that here the Duistermaat-Heckman function is piecewise polynomial.

4. Suzuki-Takakura

In 2008, Suzuki and Takakura [ST] gave a result about the symplectic volume of an N-fold reduced product $\mathcal{M}(\underline{\xi})$ for $G = SU(3)$ and $N \geq 3$, with $\underline{\xi}$ lying in a discrete set. from geometric invariant theory and representation theory).

In particular, their result for $N = 3$ is as follows.

Definition: A 6-tuple $(I_1, \cdots, I_6)$, where each I_i is a subset of $\{1, 2, 3\}$, is called a 6-partition of $\{1, 2, 3\}$, if $I_1 \cup \cdots \cup I_6 = \{1, 2, 3\}$ and $I_i \cap I_j = \emptyset$ if $i \neq j$.

- Now let $\xi_i = (\ell_i - m_i)\omega_1 + m_i\omega_2 \in \Lambda_+$ with $\ell_i > m_i > 0$, $\ell_i, m_i \in \mathbb{Z}$, for $i \in \{1, 2, 3\}$.
- Let $L := \ell_1 + \ell_2 + \ell_3$ and $M := m_1 + m_2 + m_3$.
- They assume the following condition (in order to apply GIT techniques): $L + M$ is divisible by 3.
- **Definition:** For any $I \subset \{1, 2, 3\}$, define

$$\ell_I = \sum_{i \in I} \ell_i, \quad m_I = \sum_{i \in I} m_i. \tag{34}$$

If I and J are disjoint subsets of $\{1,2,3\}$, define

$$\ell_{I,J} = \ell_I + \ell_J = \sum_{i \in I \cup J} \ell_i, \quad m_{I,J} = m_I + m_J = \sum_{i \in I \cup J} m_i. \tag{35}$$

Let $\underline{\xi} = (\xi_1, \xi_2, \xi_3)$.

- **Definition:** Denote by $\mathbf{I}_{\underline{\xi}}$ the set of all 6-partitions $(I_1, \cdots, I_6)$ such that

$$\ell_{I_1,I_2} + m_{I_4,I_5} < \frac{1}{3}(L+M), \quad \ell_{I_3,I_4} + m_{I_6,I_1} < \frac{1}{3}(L+M), \tag{36}$$

and denote by $\mathbf{J}_{\underline{\xi}}$ the set of all 6-partitions $(I_1, \cdots, I_6)$ such that

$$\ell_{I_3,I_4} + m_{I_6,I_1} > \frac{1}{3}(L+M), \quad \ell_{I_5,I_6} + m_{I_2,I_3} > \frac{1}{3}(L+M). \tag{37}$$

- Notice that $\mathbf{I}_{\underline{\xi}}$ and $\mathbf{J}_{\underline{\xi}}$ are disjoint for any $\underline{\xi}$.
- **Definition:** Define the functions $A_{\underline{\xi}} : \mathbf{I}_{\underline{\xi}} \to \mathbb{R}$ and $B_{\underline{\xi}} : \mathbf{J}_{\underline{\xi}} \to \mathbb{R}$ as follows:

$$A_{\underline{\xi}}(I_1, \cdots, I_6) := \frac{-(-1)^{|I_1|+|I_3|+|I_5|}}{6}(\frac{L+M}{3} - \ell_{I_1,I_2} - m_{I_4,I_5}), \tag{38}$$

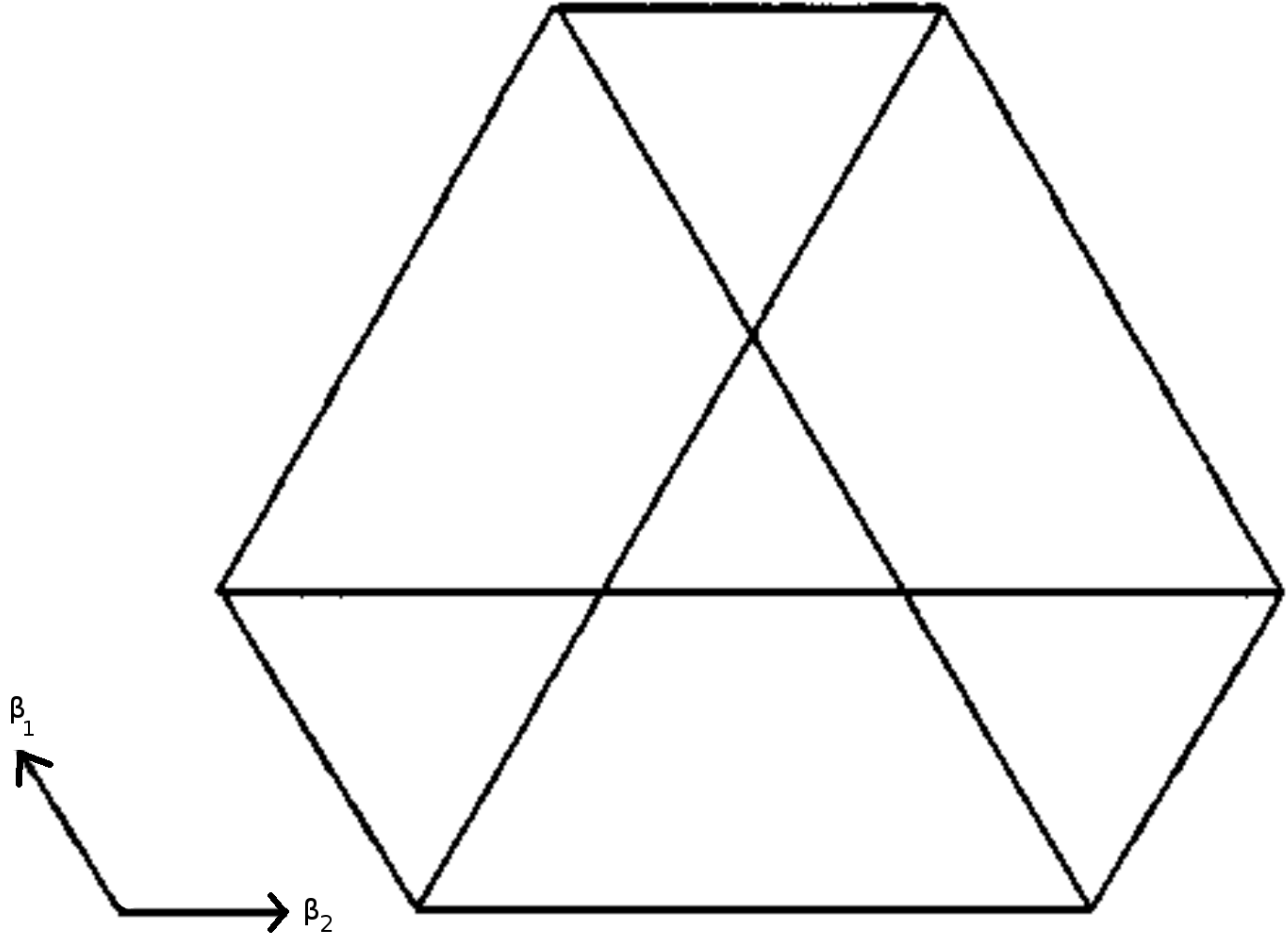
$$B_{\underline{\xi}}(I_1, \cdots, I_6) := \frac{-(-1)^{|I_1|+|I_3|+|I_5|}}{6}(\ell_{I_5,I_6} + m_{I_2,I_3} - \frac{L+M}{3}). \tag{39}$$

- Then Suzuki and Takakura conclude that the symplectic volume of $\mathcal{M}(\underline{\xi})$ is given by the following formula:

- **Theorem:**

$$\mathcal{V}(\underline{\xi}) = \sum_{(I_1,\cdots,I_6)\in\mathbf{I}_{\underline{\xi}}} A_{\underline{\xi}}(I_1,\cdots,I_6) + \sum_{(I_1,\cdots,I_6)\in\mathbf{J}_{\underline{\xi}}} B_{\underline{\xi}}(I_1,\cdots,I_6). \tag{40}$$

- Since both our formula and the formula of Suzuki and Takakura describe the symplectic volume for the same object, they should agree. There is numerical evidence that they indeed agree.



β_1
β_2