Two-dimensional Stochastic Interface Growth

Fabio Toninelli

CNRS and Université Lyon 1

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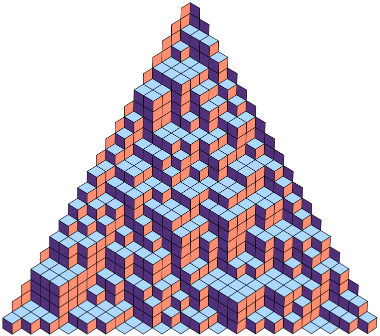
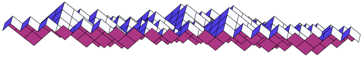
Random discrete interfaces and growth



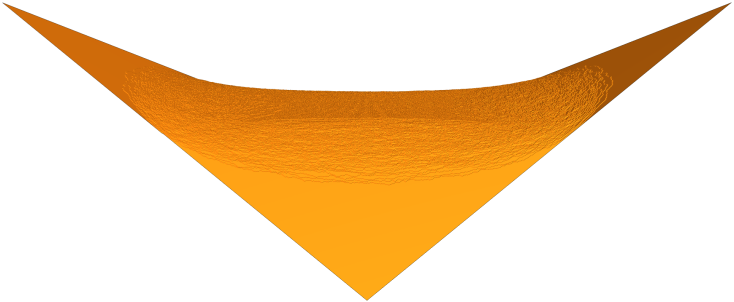


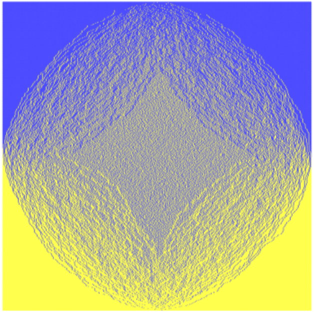
- 2d discrete interfaces $\Longrightarrow$ random tilings, dimer model
- Stochastic growth (random deposition).
  Large scales $\Longrightarrow$ non-linear PDEs, stochastic PDEs, ...
- An interesting story: Wolf's conjecture on universality classes of 2d interface growth

Random discrete interfaces and growth

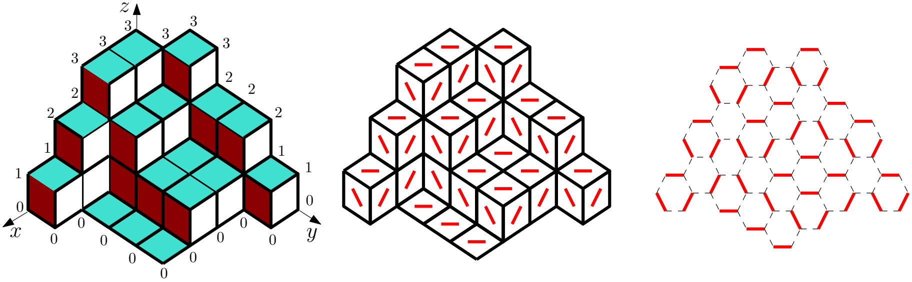
Links with:

- macroscopic shapes
- facet singularities
- massless Gaussian field (GFF)





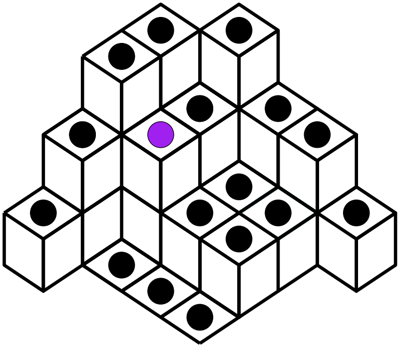
Interfaces, tilings & dimers

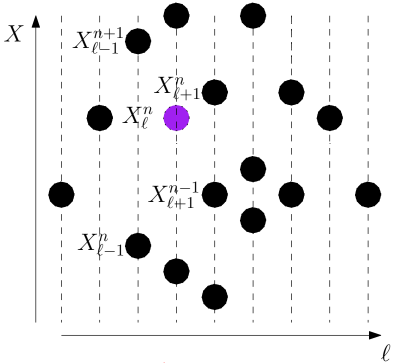


- Discrete monotone interface
- Lozenge tiling of the plane
- Dimer model (perfect matching of planar bipartite graph)

Link with spin systems: ground state of 3d Ising model

Lozenge tiling $\Leftrightarrow$ Interlaced particle system



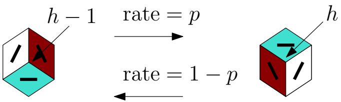


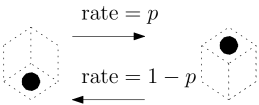
$$X^{n-1}_{\ell+1} < X^{n}_{\ell} < X^{n}_{\ell+1}$$

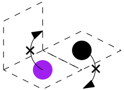
The whole interface/dimer/lozenge picture is still there

A stochastic deposition model

Continuous-time Markov process. Updates:







Jumps respect interlacing conditions

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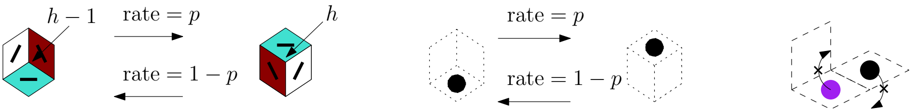
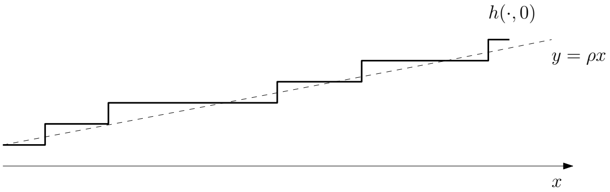


Jumps respect interlacing conditions

- symmetric case $p = 1/2$: uniform measure is stationary & reversible
- $p \neq 1/2$: growth model, irreversibility. Interesting in infinite volume (or with periodic boundary conditions)
- equivalent to zero temperature Glauber dynamics of 3d Ising
  $p \leftrightarrow$ magnetic field

Interface growth: phenomenological picture

Speed of growth $v = v(\rho)$: asymptotic growth rate for interface of slope $\rho \in \mathbb{R}^d$ (for us, $d = 2$)



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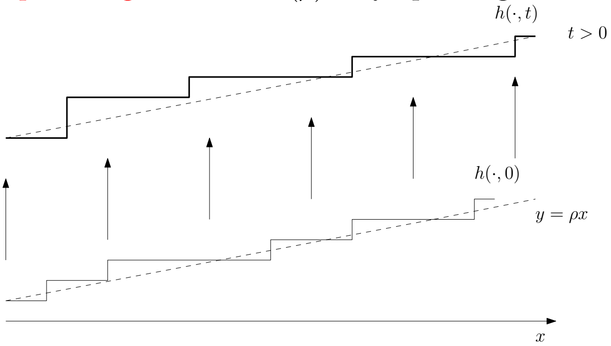


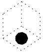
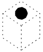
$$v(\rho) = \lim_{t\to\infty} \frac{h(x,t)-h(x,0)}{t}$$

Interface growth: phenomenological picture

- As $t \to \infty$, law of gradients

$$\nabla h \equiv (h(x+\hat{e}_i) - h(x)), \quad x \in \mathbb{Z}^d, i = 1, \dots, d$$

should tend to limit stationary, non-reversible measure π_ρ

E. g. $v(\rho) = p \times \pi_\rho($  $) - (1-p) \times \pi_\rho($  $)$

- Roughness exponent α: at large distances

$$\sqrt{\mathrm{Var}_{\pi_\rho}(h(x) - h(y))} \sim c_1 + c_2 |x-y|^{\alpha}$$

- Growth exponent β: at large times,

$$\sqrt{\mathrm{Var}(h(x,t) - h(x,0))} \sim c_3 + c_4 t^{\beta}$$

Fluctuation field and link with the KPZ equation

Heuristics: large-scales behavior of fluctuations $\rightsquigarrow$ Kardar-Parisi-Zhang equation

relaxes large fluctuations — tunes strength of non-linearity. Useful in perturbation theory

$$\partial_t h(x,t) = \Delta h(x,t) + \lambda (\nabla h(x,t), H \nabla h(x,t)) + \xi_{\text{smooth}}(x,t)$$

$d \times d$ symmetric matrix — smoothed space-time white noise

Quadratic non-linearity from second-order Taylor expansion of hydrodynamic PDE.

$$H = D^2 v(\rho) \qquad \text{(Hessian of speed of growth)}$$

Fluctuation field and link with the KPZ equation

$$\partial_t h(x,t) = \Delta h(x,t) + \lambda(\nabla h(x,t), H\nabla h(x,t)) + \xi_{\text{smooth}}(x,t)$$

- Linear case ($\lambda = 0$): Edwards-Wilkinson (EW) equation.
  Stationary state: massless Gaussian field.

$$\alpha_{EW} = (2-d)/2, \qquad \beta_{EW} = (2-d)/4.$$

- $d = 1$: KPZ '86 predicted relevance of non-linearity.

$$\beta = \frac{1}{3} \neq \beta_{EW}$$

  Confirmed by exact solutions (1-d KPZ universality class: universal non-Gaussian limit laws, ...)

- $d \geq 3$: predicted irrelevance of small non-linearity, transition at λ_c.
  $\Rightarrow$ see Magnen-Unterberger '17, Gu-Ryzhik-Zeitouni '17 for $\lambda \ll 1$

The critical dimension $d = 2$ and Wolf's conjecture

$$\partial_t h(x,t) = \Delta h(x,t) + \lambda(\nabla h(x,t), H\nabla h(x,t)) + \xi_{\text{smooth}}(x,t)$$

One-loop perturbative (in λ) Renormalization-Group analysis (D. Wolf '91):

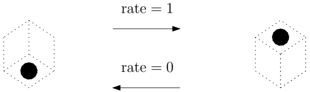
- if $\det(H) > 0$, non-linearity relevant, $\alpha \neq \alpha_{EW}, \beta \neq \beta_{EW}$;
- if $\det(H) \leq 0$, small non-linearity irrelevant. EW Universality class.

Conjecture: Two universality classes:

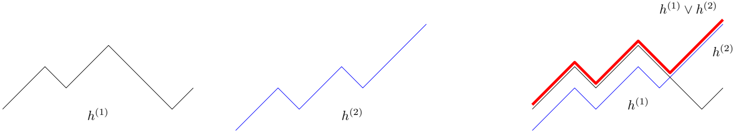
- Anisotropic KPZ (AKPZ) class: $\det(D^2 v(\rho)) \leq 0$.
  Large-scale fixed point: EW equation. $\alpha_{\text{AKPZ}} = 0, \beta_{\text{AKPZ}} = 0$.
- KPZ class: $\det(D^2 v(\rho)) > 0$. $\alpha_{\text{KPZ}} \neq 0, \beta_{\text{KPZ}} \neq 0$.

Numerics (Halpin-Healy et al.): in KPZ class, universal exponents $\alpha_{\text{KPZ}} \approx 0.39..., \beta_{\text{KPZ}} \approx 0.24...$.

Back to the deposition process



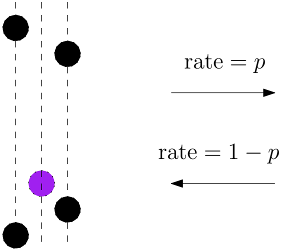
Envelope property: $h(t=0) = h^{(1)} \vee h^{(2)} \Longrightarrow h(t) = h^{(1)}(t) \vee h^{(2)}(t)$

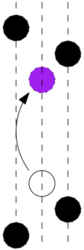


Then, superadditivity argument (T. Seppäläinen, F. Rezakhanlou) implies that $v(\cdot)$ exists and is convex.

Natural candidate for KPZ class. No math results on stationary states or critical exponents $\alpha_{\mathrm{KPZ}}, \beta_{\mathrm{KPZ}}$

A long-jump variant

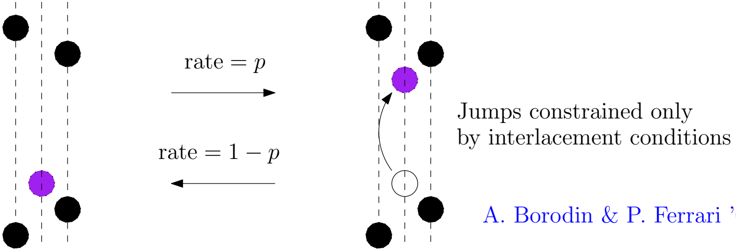




Jumps constrained only
by interlacement conditions

A. Borodin & P. Ferrari '08

A long-jump variant



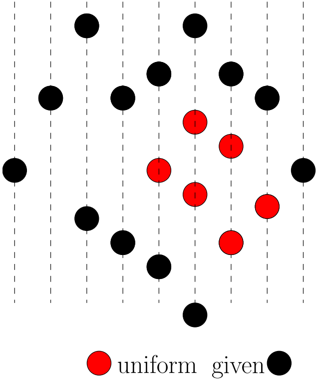
A. Borodin & P. Ferrari '08

Should the universality class change? not obvious a priori.

In fact, it does change

AKPZ signature

Theorem (F.T., 15) Stationary states π_ρ are "locally uniform"



- Stationary states free-fermionic (determinantal correlations)
- Roughness exponent : $\alpha = 0$
  logarithmic fluctuations,
  scaling to massless Gaussian field
- Growth exponent $\beta = 0$
  $\mathrm{Var}_{\pi_\rho}(h(x,t) - h(x,0)) \stackrel{t\to\infty}{=} O(\log t)$

Speed and Hydrodynamic limit

Theorem (M. Legras, F.T. '17)
If

$$\lim_{\epsilon\to 0} \epsilon h(\epsilon^{-1}x, t=0) = \phi_0(x), \quad \forall x \in \mathbb{R}^2$$

with $\phi_0(\cdot)$ convex, then

$$\lim_{\epsilon\to 0} \epsilon h(\epsilon^{-1}x, \epsilon^{-1}t) = \phi(x,t), \quad t > 0$$

(with high probability as $\epsilon \to 0$) where ϕ solves

$$\begin{cases} \partial_t \phi(x,t) = v(\nabla\phi(x,t)) \\ \phi(x,0) = \phi_0(x). \end{cases}$$

Speed of growth $v(\rho)$: explicit and $\det D^2 v(\rho) < 0$

Comments on hydrodynamic equation

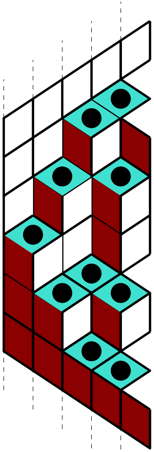
- Non-linear Hamilton-Jacobi equation $\Rightarrow$ singularities in finite time
- Physically relevant solution: viscosity solution.

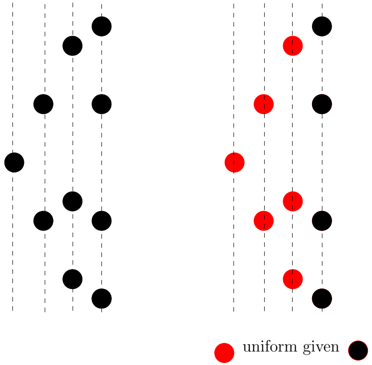
$$v(\nabla\phi) \mapsto v(\nabla\phi) + \epsilon\Delta\phi, \quad \epsilon \to 0^+$$

- $v(\cdot)$ non convex $\Rightarrow$ no variational formula (like "minimal action") for viscosity solution.
  For convex profile, variational formula.
- Technical difficulty: long jumps, possible pathologies
  (tools: from works of T. Seppäläinen)

Previous results on the model

Theorem (A. Borodin, P. Ferrari '08)
For "triangular-array Gibbs-type initial conditions", hydrodynamic limit and central limit theorem on scale $\sqrt{\log t}$.



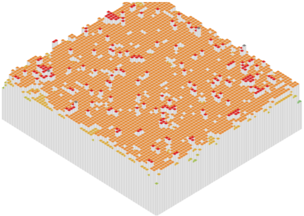


Smooth phases and singularities of $v(\cdot)$

For equilibrium 2d discrete interface models, smooth (or "rigid") (as opposed to: rough) phases at special slopes

Exponential decay of correlations, no fluctuation growth:

$$\sup_x \mathrm{Var}(h(x) - h(0)) < \infty,$$



E.g. SOS model at low temperature; dimers ("gas phases"),...

Smooth phases and singularities of $v(\cdot)$

Questions:

- AKPZ growth models with smooth stationary states?
- We implicitly assumed that speed $v(\cdot)$ is differentiable ($H = D^2 v$ in KPZ Eq.)

  What if it is not?

  - Still Edwards-Wilkinson behavior?
  - Link with smooth stationary states?

An AKPZ model with a smooth phase

Together with S. Chhita, we studied a growth model where:

- height function is $h : \mathbb{Z}^2 \ni x \mapsto h(x) \in \mathbb{Z}$
- Growth process in discrete time: $h_0(\cdot), h_1(\cdot), h_2(\cdot), \ldots$
- Local update rule: $h_n(x) \to h_{n+1}(x) \rightsquigarrow$ random function of neighboring values

$$h_n(y), \quad |y - x| = 1$$

- Stationary states π_ρ of ∇h are
  - logarithmically rough for $\rho \neq 0$, i.e. $\mathrm{Var}_{\pi_\rho}(h(x) - h(y)) \sim \log|x - y|$
  - smooth for $\rho = 0$, i.e. $\mathrm{Var}_{\pi_0}(h(x) - h(y)) = O(1)$

For experts: dynamics is domino-shuffling algorithm with 2-periodic weights (J. Propp)

An AKPZ model with a smooth phase

Theorem (S. Chhita, F.T. '18)
For $\rho \neq 0$, AKPZ signature:

- Logarithmic growth of fluctuations:

$$\mathrm{Var}_{\pi_\rho}(h(x,t) - h(x,0)) = O(\log t)$$

- Twice differentiable speed and

$$\det(D^2 v(\rho)) < 0.$$

For $\rho = 0$, new picture:

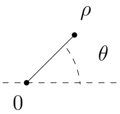
- bounded fluctuations:

$$\mathrm{Var}_{\pi_0}(h(x,t) - h(x,0)) = O(1)$$

- Non-differentiability of $v(\cdot)$ at 0

Smooth phases, facets and singularities of $v(\cdot)$

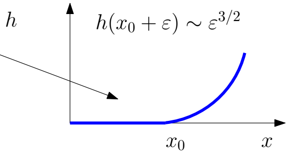
Non-differentiability related to facets of macroscopic shapes



related to "facet singularities"

$$v(\rho) \overset{\rho\to 0}{\approx} |\rho| f_1(\theta) + |\rho|^3 f_2(\theta)$$

non-differentiability



Pokrovsky-Talapov law

Smooth phases, facets and singularities of $v(\cdot)$

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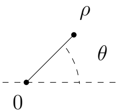

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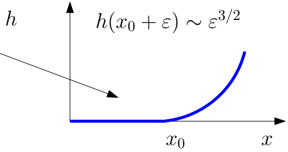

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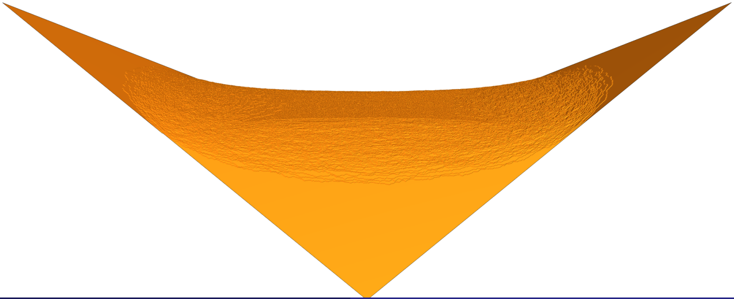
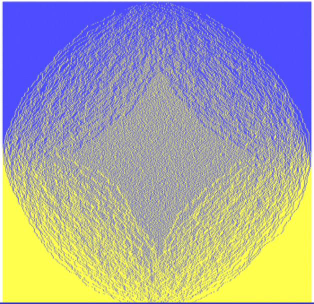




A more general AKPZ class

- A more general class of interlaced-particle dynamics that includes both previous examples (Borodin & Ferrari '08)
- Fluctuation & hydrodynamic results have been extended to this context
- Puzzling points:
  - explicit computation of speed $\Longrightarrow \det(D^2 v) < 0$ without clear connection to Wolf's heuristics.
  - speed is harmonic w.r.t. suitable complex structure

Any pattern behind?

AKPZ growth and Euler-Lagrange equation

A geometric argument behind $\det(D^2 v(\rho)) \leq 0$ for AKPZ models (A. Borodin, F.T., '18)

- Common feature of most known AKPZ growth models: stationary, non-reversible Gibbs measures π_ρ:

$$\nabla h(t=0) \sim \pi_\rho \Longrightarrow \nabla h(t) \sim \pi_\rho$$

- Gibbs states π: probability measures such that

$$\text{law of } h(x) \text{ given } h|_{\mathbb{Z}^2 \setminus \{x\}} \text{ depends only on } \{h(y)\}_{|y-x|=1}.$$

In many examples, π_ρ locally uniform, free-fermionic

AKPZ growth and Euler-Lagrange equation

- $\exists$ continuum of non-translation-invariant Gibbs measures and

$$\nabla h(t=0) \sim \pi^{(0)} \Longrightarrow \nabla h(t) \sim \pi^{(t)}.$$

- Macroscopically, typical height profile sampled from Gibbs state is minimizer ϕ of surface tension functional

$$\int_{\mathbb{R}^2} \sigma(\nabla\phi)dx$$

with $\sigma(\cdot)$ convex, i.e. solution of Euler-Lagrange equation

$$\sum_{i,j=1}^{2} \sigma_{ij}(\nabla\phi)\partial^2_{x_i x_j}\phi = 0, \qquad (\sigma_{ij}(\rho) := \partial^2_{\rho_i\rho_j}\sigma(\rho)).$$

AKPZ growth and Euler-Lagrange equation

Preservation of Gibbs property $\Longrightarrow$ hydrodynamic PDE

$$\partial_t \phi = v(\nabla \phi)$$

preserves solutions of Euler-Lagrange:

$$\phi(t=0) \text{ solves Euler-Lagrange} \Longrightarrow \phi(t) \text{ does too}$$

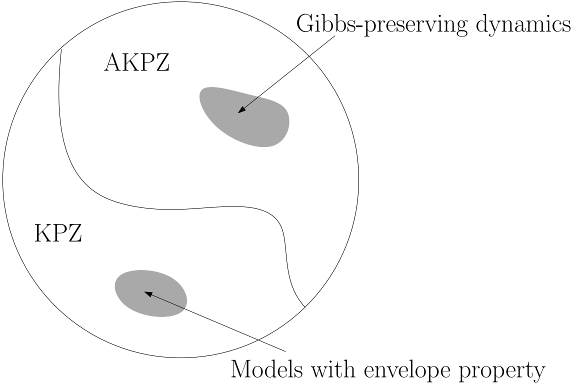
Theorem (A. Borodin, F.T.) This gives a non-linear relation between $D^2 v$ and $D^2 \sigma$, that implies $\det(D^2 v) \leq 0$.

For dimer models, solutions of Euler-Lagrange parametrized by complex variable $z = z(\nabla \phi)$ (R. Kenyon & A. Okounkov '07)

Theorem (A. Borodin, F.T.) Hydrodynamic PDE preserves Euler-Lagrange equation $\Longleftrightarrow$ speed $v(\cdot)$ is harmonic function of z.

Where do we stand?

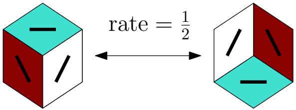
The world of 2d stochastic growth processes



Is any of the two classes “generic”?

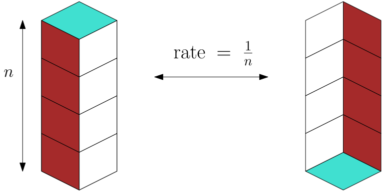
How to guess universality class from symmetries of generator?

Things that were left out



reversible dynamics

- Bounds $O(L^{2+\epsilon})$ on mixing time in finite $L \times L$ domain (P. Caputo, B. Laslier, F. Martinelli, F.T.)
- Convergence to non-linear parabolic PDE for long-jump symmetric dynamics (B. Laslier, F.T. '17)



Summary

We discussed Wolf's conjecture on universality classes of 2d stochastic interface growth.

For a class of AKPZ growth models:

- hydrodynamic limits
- logarithmic bounds on fluctuation growth, $\alpha_{AKPZ} = \beta_{AKPZ} = 0$
- singularities of $v(\cdot) \longleftrightarrow$ smooth phases, facets
- origin of $\det D^2 v \leq 0$: preservation in time of Gibbs property

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Open problem:

- Full convergence to Edwards-Wilkinson fixed point? (proven in limiting regimes: A. Borodin, I. Corwin & F.T. '17, A. Borodin, I. Corwin & P. Ferrari '17)

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Thanks!