Fractal uncertainty principle and quantum chaos

Semyon Dyatlov (UC Berkeley/MIT)

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Overview

- This talk presents two recent results in quantum chaos
- Central ingredient: fractal uncertainty principle (FUP)

No function can be localized
in both position and frequency
near a fractal set

- Using tools from
  - Microlocal analysis (classical/quantum correspondence)
  - Hyperbolic dynamics (classical chaos)
  - Fractal geometry
  - Harmonic analysis

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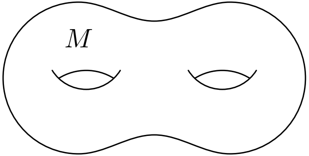
First result: lower bound on mass

- (M, g) compact hyperbolic surface (Gauss curvature $\equiv -1$)
- Geodesic flow on M: a standard model of classical chaos (perturbations diverge exponentially from the original geodesic)
- Eigenfunctions of the Laplacian $-\Delta_g$ studied by quantum chaos



$$(-\Delta_g - \lambda^2)u = 0, \quad \|u\|_{L^2} = 1$$

Theorem 1 [D–Jin '17, using D–Zahl '15 and Bourgain–D '16]

Let $\Omega \subset M$ be a nonempty open set. Then there exists c depending on M, Ω but not on λ s.t.

$$\|u\|_{L^2(\Omega)} \geq c > 0$$

For bounded λ this follows from unique continuation principle

The new result is in the high frequency limit $\lambda \to \infty$

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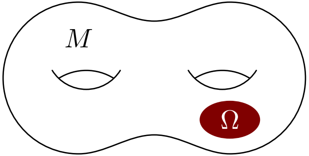

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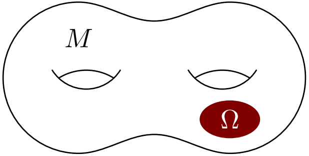

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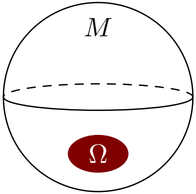

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The chaotic nature of geodesic flow is important

For example, Theorem 1 is false if M is the round sphere

Theorem 1

Let M be a hyperbolic surface and $\Omega \subset M$ a nonempty open set. Then there exists $c_\Omega > 0$ s.t.

$$(-\Delta_g - \lambda^2)u = 0 \quad \Longrightarrow \quad \|u\|_{L^2(\Omega)} \geq c_\Omega \|u\|_{L^2(M)}$$

Application to control theory (using standard techniques e.g. Burq–Zworski '04, '12):

Theorem 2 [Jin '17]

Fix $T > 0$ and nonempty open $\Omega \subset M$. Then there exists $C = C(T, \Omega)$ such that

$$\|f\|^2_{L^2(M)} \leq C \int_0^T \int_\Omega |e^{it\Delta_g} f(x)|^2 \, dxdt \quad \text{for all } f \in L^2(M)$$

Control by any nonempty open set previously known only for flat tori: Haraux '89, Jaffard '90

Work in progress

- Datchev–Jin: an estimate on c_Ω in terms of Ω (using Jin–Zhang '17)
- D–Jin–Nonnenmacher: Theorems 1 and 2 for surfaces of variable negative curvature

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Weak limits of eigenfunctions

Theorem 1 arose from trying to understand high frequency sequences of eigenfunctions

$$(-\Delta_g - \lambda_j^2)u_j = 0, \quad \|u_j\|_{L^2} = 1, \quad \lambda_j \to \infty$$

in terms of weak limit: probability measure μ on M such that $u_j \to \mu$ in the following sense

$$\int_M a(x)|u_j(x)|^2 \, d\,\mathrm{vol}_g(x) \to \int_M a \, d\mu \quad \text{for all} \quad a \in C^\infty(M)$$

Theorem 1 $\Rightarrow$ for hyperbolic surfaces, every μ has $\operatorname{supp}\mu = M$: 'no whitespace'

A (much) stronger property is equidistribution: $\mu = d\,\mathrm{vol}_g$

- Quantum ergodicity: geodesic flow is chaotic $\Rightarrow$ most eigenfunctions equidistribute Shnirelman '74, Zelditch '87, Colin de Verdière '85 . . . Zelditch–Zworski '96
- QUE conjecture [Rudnick–Sarnak '94]: all eigenfunctions equidistribute for strongly chaotic systems. Only proved in arithmetic situations: Lindenstrauss '06
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Pictures of eigenfunctions (courtesy of Alex Barnett)

One can also study Dirichlet eigenfunctions on a domain with boundary

The geodesic flow is replaced by the billiard ball flow

Completely integrable

Whitespace in the center (easy)

Mildly chaotic

Whitespace on the sides (conj.)
Lack of equidistribution [Hassell '10]

Strongly chaotic

No whitespace (conj., similar to Theorem 1)
Equidistribution (conj., QUE)

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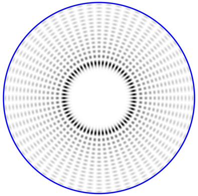

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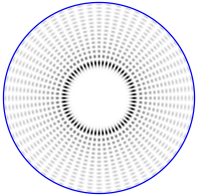

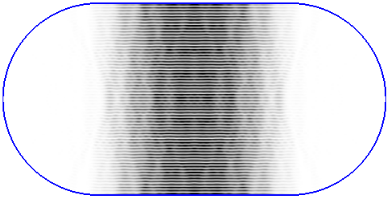

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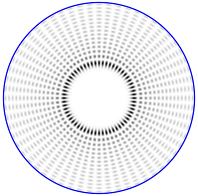

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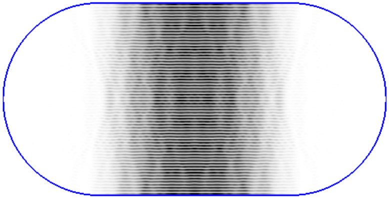

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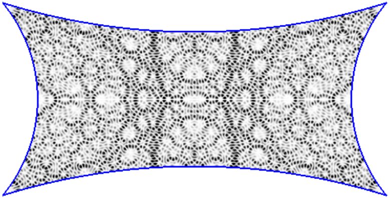

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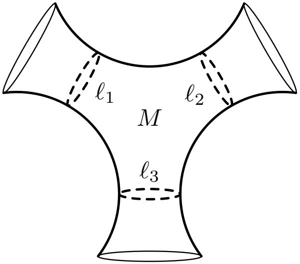
Second result: spectral gaps for noncompact hyperbolic surfaces

(M, g) convex co-compact hyperbolic surface



Pictures of resonances
(by David Borthwick and Tobias Weich)

Resonances: zeroes of the Selberg zeta function

$$Z_M(s) = \prod_{\ell \in \mathcal{L}_M} \prod_{k=0}^{\infty} \left(1 - e^{-(s+k)\ell}\right)$$

where $\mathcal{L}_M = \{\text{lengths of primitive closed geodesics}\}$

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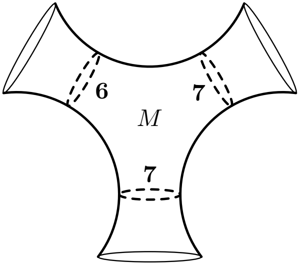

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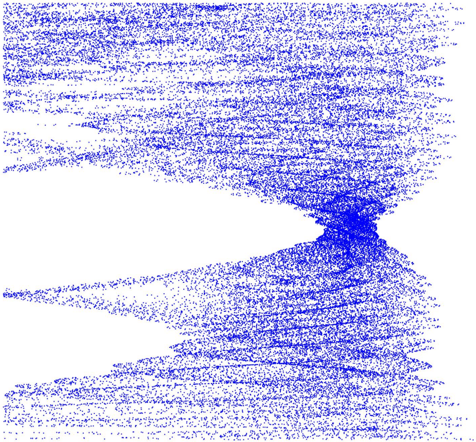

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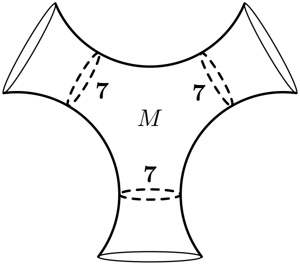

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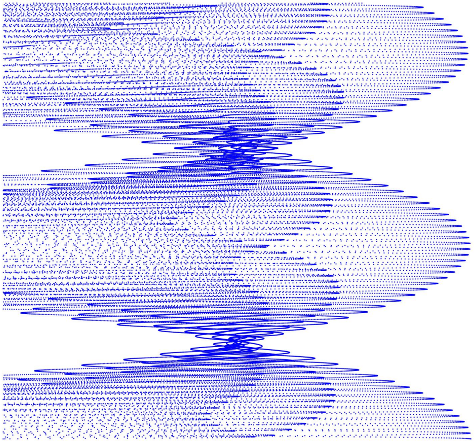

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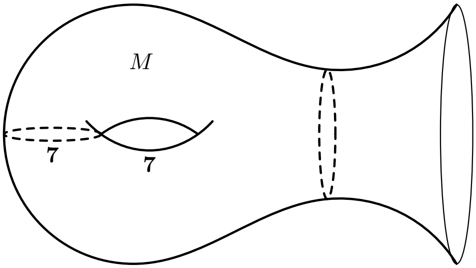

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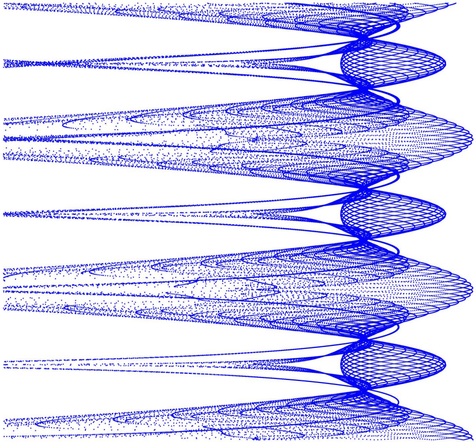

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Theorem 3 [D–Zahl '15, Bourgain–D '16, D–Zworski '17]

Let M be a convex co-compact hyperbolic surface. Then there exists an essential spectral gap of size $\beta = \beta(M) > 0$, namely M has only finitely many resonances s with $\operatorname{Re} s > \frac{1}{2} - \beta$

- Previously known only for 'thinner half' of surfaces: Patterson '76, Sullivan '79, Naud '05
- Gap for 'thin' open systems: Ikawa '88, Gaspard–Rice '89, Nonnenmacher–Zworski '09
- Applications to exponential decay for waves and Strichartz estimates: Wang '17
- Conjecture: every strongly chaotic scattering system has a spectral gap
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- Density results supporting stronger conjecture: Naud '14, D '15

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Main ingredient: fractal uncertainty principle (FUP)

Definition

Fix $\nu > 0$. A set $X \subset \mathbb{R}$ is ν-porous up to scale $\hbar$ if for each interval $I \subset R$ of length $\hbar \le |I| \le 1$, there is an interval $J \subset I$, $|J| = \nu|I|$, $J \cap X = \emptyset$

Theorem 4 [Bourgain–D '16]

Let $\hbar \ll 1$ and X, Y be ν-porous up to scale $\hbar$. Then there exists $\beta = \beta(\nu) > 0$:

$$f \in L^2(\mathbb{R}), \quad \operatorname{supp} \hat{f} \subset \hbar^{-1} \cdot Y \quad \Longrightarrow \quad \|\mathbf{1}_X f\|_{L^2(\mathbb{R})} \le C\hbar^{\beta} \|f\|_{L^2(\mathbb{R})}$$

"Cannot concentrate in both position and frequency on a fractal set"

Tools: Beurling–Malliavin theorem, iteration on scales...

Recent progress: Jin–Zhang '17 (quantitative version),
Han–Schlag '18 (some higher dimensional cases)

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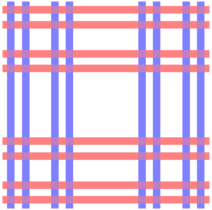

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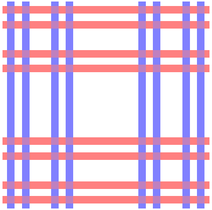

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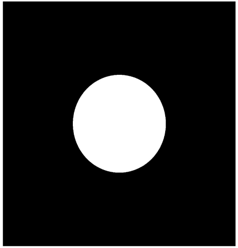
How do fractal sets appear?

- $\varphi_t : S^*M \to S^*M$ the geodesic flow on (M, g)
- $U \subset S^*M$ open nonempty set, called the **hole**
- The fractal sets to which FUP is applied arise from geodesics missing the hole:

$$\Gamma_{\pm}(T) := \{\rho \in S^*M \mid \varphi_{\mp t}(\rho) \notin U \text{ for all } t \in [0, T]\}$$



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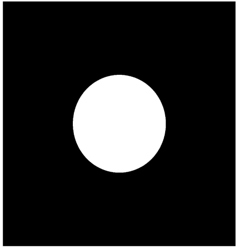

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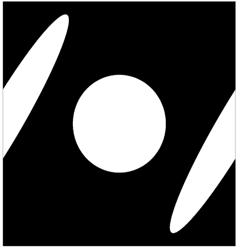
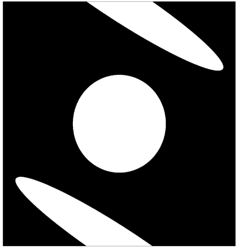


$\Gamma_-(T),\ T = 1$

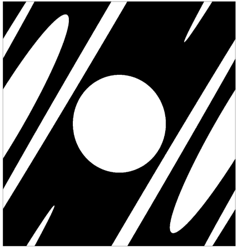


$\Gamma_+(T),\ T = 1$

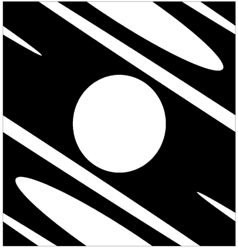
How do fractal sets appear?

- $\varphi_t : S^*M \to S^*M$ the geodesic flow on (M, g)
- $U \subset S^*M$ open nonempty set, called the **hole**
- The fractal sets to which FUP is applied arise from geodesics missing the hole:

$$\Gamma_\pm(T) := \{\rho \in S^*M \mid \varphi_{\mp t}(\rho) \notin U \text{ for all } t \in [0, T]\}$$



$\Gamma_-(T)$, $T = 2$



$\Gamma_+(T)$, $T = 2$

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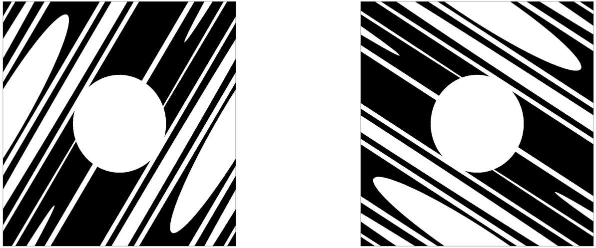

$\Gamma_-(T)$, $T = 3$ $\Gamma_+(T)$, $T = 3$

How do fractal sets appear?

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- The fractal sets to which FUP is applied arise from geodesics missing the hole:

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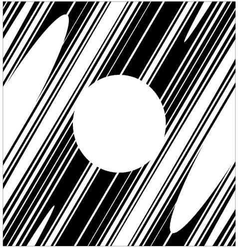
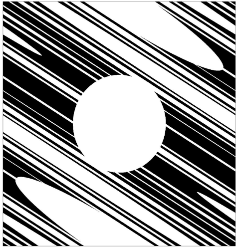


$\Gamma_-(T)$, $T = 4$

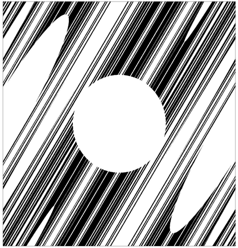


$\Gamma_+(T)$, $T = 4$

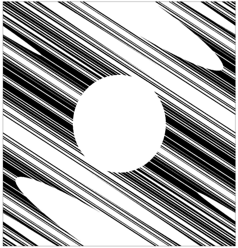
How do fractal sets appear?

- $\varphi_t : S^*M \to S^*M$ the geodesic flow on (M, g)
- $U \subset S^*M$ open nonempty set, called the **hole**
- The fractal sets to which FUP is applied arise from geodesics missing the hole:

$$\Gamma_\pm(T) := \{\rho \in S^*M \mid \varphi_{\mp t}(\rho) \notin U \text{ for all } t \in [0, T]\}$$



$\Gamma_-(T)$, $T = 5$



$\Gamma_+(T)$, $T = 5$

Thank you for your attention!