Emergence of non-ergodic dynamic

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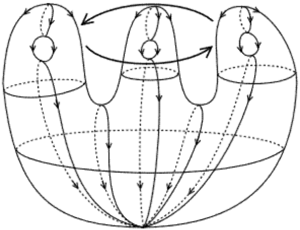
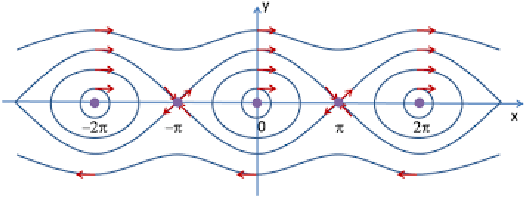
For some systems, this problem is very simple, for most it is not.

Systems for which it is easy

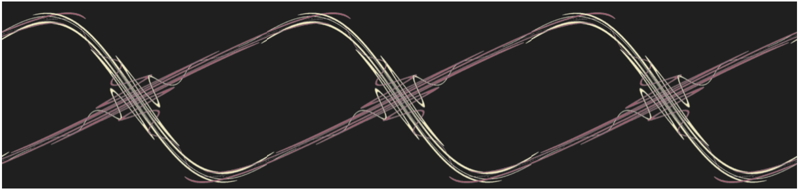
- this is easy for Morse-Smale dynamics:



- This is easy for some integrable systems.

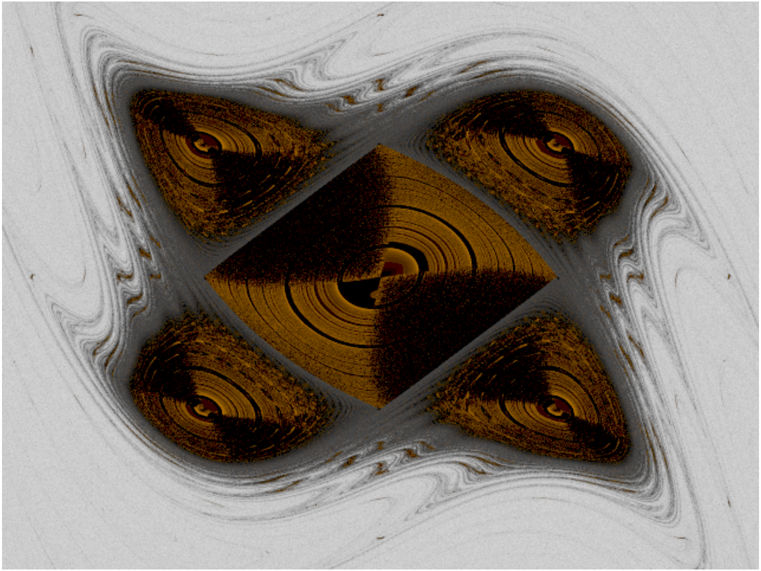


Systems for which it is not easy

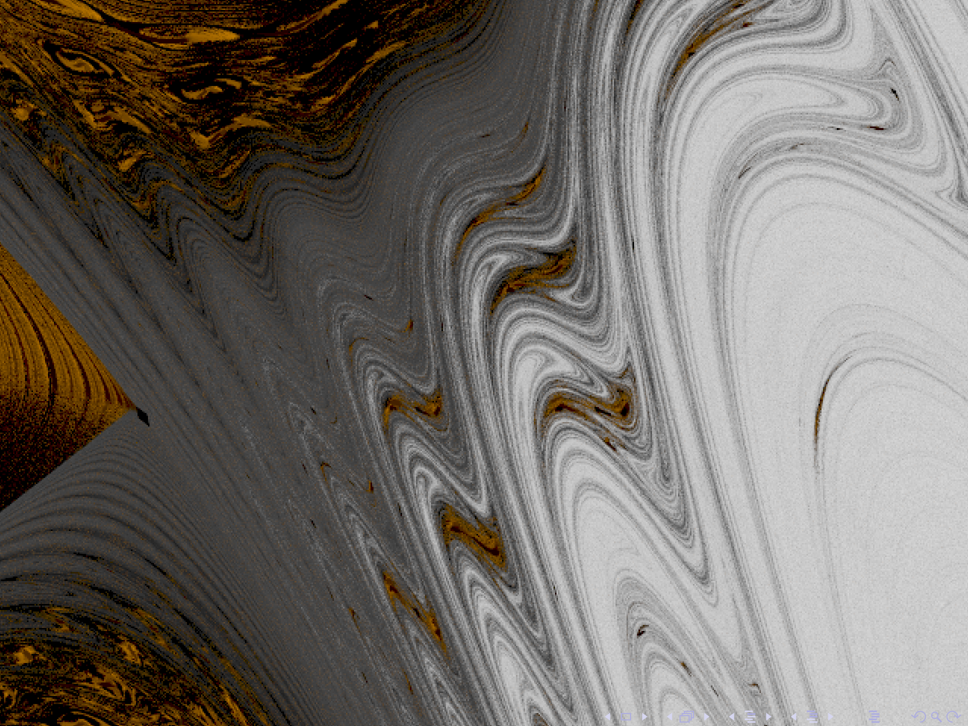


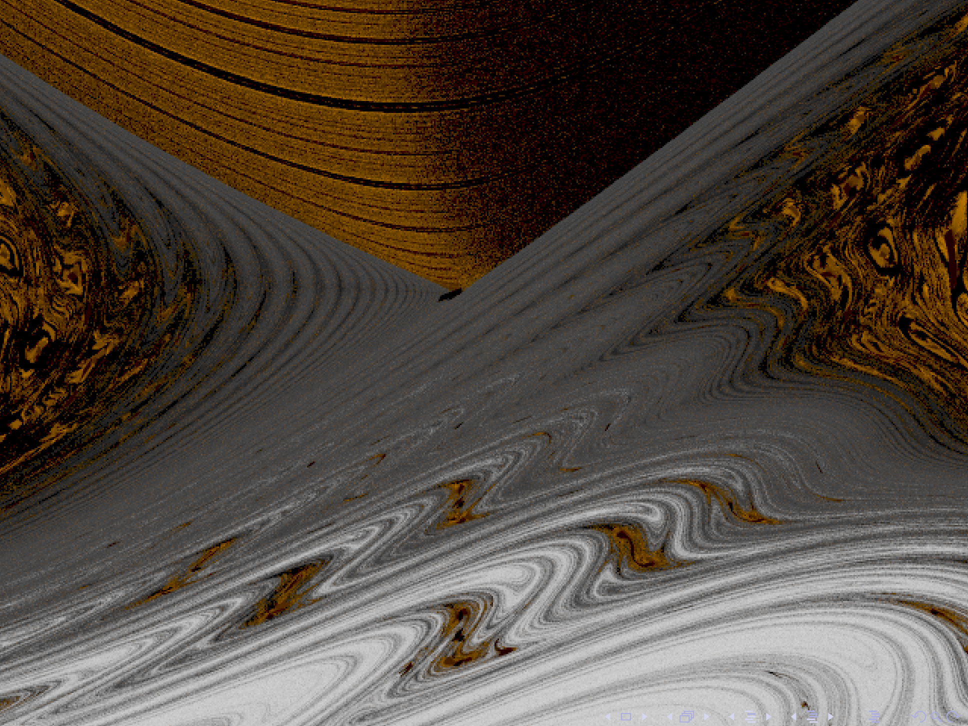
First discovered by Poincaré, see also Hadamard, Kolmogorov, Anosov, Sinai, Smale etc...

Standard map (phase space)



Let's zoon in!





How to describe this?

Entropy as a quantificator of the complexity of those systems

Topological entropy: Let $H_{top}(n)$ be the number of points necessarily to shadow the n first iterates of all the points. Put:

$$h_{top} := \lim \frac{1}{n} \log H_{top}(n) .$$

Metric entropy: Let $H_{\mathrm{Leb}}(n)$ be the number of points necessarily to shadow the n first iterates of Lebesgue nearly all the points. Put:

$$h_{\mathrm{Leb}} := \lim \frac{1}{n} \log H_{\mathrm{Leb}}(n) .$$

Such a definition can be done for any measure μ instead of Leb. This defines h_μ.

Understanding those systems

To simplify one can focus on the statistics behavior of the orbits of such dynamics.

The statistical behavior of the orbit of x for a dynamics f is given by the sequence of the n^{th}-Birkhoff averages:

$$\delta_f^n(x) := \frac{1}{n}\sum_{i=0}^{n-1} \delta_{f^i(x)} \ .$$

We denote by $\delta_f^\infty(x)$ the set of cluster values of this sequence.

Question

Does the statistical behavior of a typical dynamical systems is easy to understand for Lebesgue nearly all the points?

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Those conjectures are satisfied among uniformly hyperbolic systems (Anosov, Sinai, Ruel, Bowen), among most partially hyperbolic systems (Pesin, Pugh, Shub, Bonatti, Viana, Avila, Crovisier, Wilkinson, B.-Carrasco, Tsujii, Pujals) and among the quadratic maps (Lyubich, conjecture of Fatou).

These conjectures assumed the following phenomena to be negligible.

Definition (Dynamics displaying Newhouse phenomena)

there exists infinitely many attracting cycles accumulating on the space of ergodic measures of a uniformly hyperbolic set.

Newhouse proved the locally Baire genericity of this phenomena among dissipative C^r-maps for $r \geq 2$. Duarte among conservative dynamics. Bonatti-Diaz for $r \geq 1$. Buzzard among holomorphic dynamics.

Definition (Phenomena Kolmogorov C^r-typical)

The phenomena occurs at every parameter of a C^r-generic family of dynamics.

The following is in opposition to the latter conjectures:

Theorem (Berger [1] [2])

Newhouse phenomena is locally Kolmogorov C^r-typical, for every $r < \infty$.

[1] Pierre Berger, Inventiones Mathematicae 2016

[2] Pierre Berger, Proceeding of the Steklov institute 2017

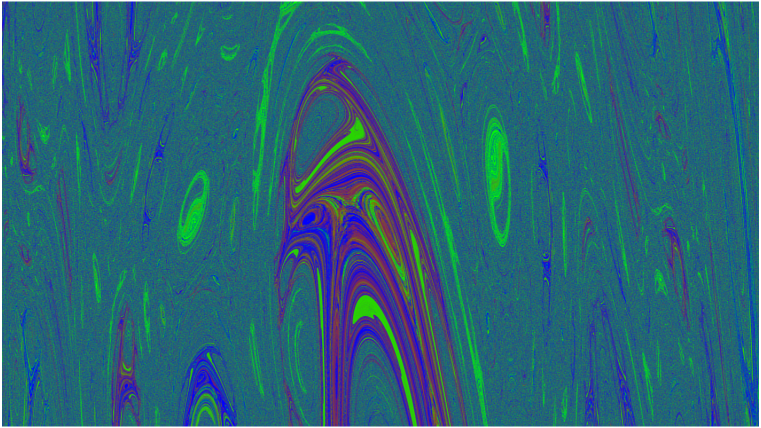

Figure: Dynamics displaying infinitely many attractors.

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How to describe their complexity?

We quantify the complexity to approximate the statistical behavior of the orbit by a finite number of probabilities. Let d be the Wasserstein distance on the space of probability measures of the manifold M.

Definition ([3])

The Emergence of a dynamics f at scale $\epsilon > 0$ is the minimal number $N = \mathcal{E}(\epsilon)$ of probabilities $(\mu_i)_{i=1}^N$ such that ϵ-nearly (Leb) every $x \in M$ has a statistical behavior which is ϵ-close to one of the measure μ_i.

- An ergodic conservative map has emergence 1.
- Newhouse phenomenon has not finite emergence.
- the identity of a d-manifold satisfies $\lim_{\epsilon\to 0} \frac{\log \mathcal{E}_{\mathrm{Leb}}(f)}{-\log \epsilon} = d$
- If KAM phenomena occurs, the emergence is at least polynomial.

Conjecture ([3])

In many categories of differentiable dynamics, a typical dynamics f displays super polynomial emergence:

$$\limsup_{\epsilon\to 0} \frac{\log \mathcal{E}_{\mathrm{Leb}}(f)}{-\log \epsilon} = \infty . \qquad \text{(Super P)}$$

[3] Pierre Berger, Proceeding of the Steklov institute 2017

Theorem (Berger–Bochi)

Let $\mathcal{U}$ be the open set of C^∞-sympletic mappings of a surface M^2 which displays an elliptic periodic point. Then a generic map $f \in \mathcal{U}$ satisfies:

$$\limsup_{\epsilon\to 0} \frac{\log\log \mathcal{E}_{\mathrm{Leb}}(f_a)(\epsilon)}{-\log\epsilon} = 2.$$

Remark

Conservative, surface mappings far from displaying an elliptic periodic point are conjecturally uniformly hyperbolic (and so stably ergodic).

Theorem (Berger–Turaev, in progress)

Let $\mathcal{U}$ be the open set of C^∞-sympletic mappings of a manifold M^{2n} which displays a totally elliptic periodic point. Then a generic family $(f_a)_a \in C^\infty(\mathbb{R}^k, \mathcal{U})$ satisfies that for every $a \in \mathbb{R}^k$:

$$\limsup_{\epsilon\to 0} \frac{\log\log \mathcal{E}_{\mathrm{Leb}}(f_a)(\epsilon)}{-\log\epsilon} = 2n.$$

This solves the conjecture in the category of Hamiltonian diffeomorphisms.

Comparing emergence and entropy.

Conjecture (Entropy)

Positive metric entropy is typical.

Theorem (Herman-Berger-Turaev)

Every C^∞-surface, conservative diffeo which displays an elliptic periodic point can be approximated to a conservative diffeomorphism with positive metric entropy.

Conjecture (Emergence)

Super polynomial emergence is typical.

Theorem (Berger, Bochi, Turaev)

For every $\infty \geq r \geq 5$, a generic C^r-surface, conservative diffeomorphism which displays an elliptic point has super exponential emergence.

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- C^∞-conservative mappings with maximal emergence and entropy zero.
- C^∞-conservative mappings with positive entropy and which are ergodic (and so trivial emergence).

Let $\mathcal{M}_f$ be the set of invariant probability measures.

Theorem (Variational Principle for entropy)

$$\sup_{\mu \in \mathcal{M}_f(X)} h_\mu(f) = h_{top}(f) \ .$$

Given $\mu \in \mathcal{M}_f$, the metric emergence $\mathcal{E}_\mu(\epsilon)$ is the minimal number N of measures $(\mu_i)_{i=1}^N$ such that $(1-\epsilon)$-μ-a.e. $x \in M$ has its statistical behavior ϵ-close to one of the measure μ_i.

The topological emergence $\mathcal{E}_{top}(f)$ is the covering number of $\mathcal{M}_e(X)$.

Theorem (Variational Principle, Berger-Bochi)

If X has box dimension d, then

$$\max_{\mu \in \mathcal{M}_f(X)} \limsup_{\epsilon \to 0} \frac{\log \log \mathcal{E}_\mu(f)(\epsilon)}{-\log \epsilon} = \limsup_{\epsilon \to 0} \frac{\log \log \mathcal{E}_{top}(f)(\epsilon)}{-\log \epsilon} \ .$$