Solvable models of quantum matter without quasiparticles

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Quasiparticles are ubiquitous:

- Fermi liquid theory of metals, insulators, semiconductors

- Theory of superconductivity (pairing of quasiparticles)

- Theory of disordered metals and insulators (diffusion and localization of quasiparticles)

- Theory of metals in one dimension (collective modes as quasiparticles)

- Theory of the fractional quantum Hall effect (quasiparticles which are `fractions' of an electron)

What are quasiparticles ?

- **Quasiparticles are additive excitations:** The low-lying excitations of the many-body system can be identified as a set $\{n_\alpha\}$ of quasiparticles with energy ε_α

$$E = \sum_\alpha n_\alpha \varepsilon_\alpha + \sum_{\alpha,\beta} F_{\alpha\beta} n_\alpha n_\beta + \ldots$$

In a lattice system of N sites, this parameterizes the energy of $\sim e^{\alpha N}$ states in terms of poly(N) numbers.

What are quasiparticles ?

- Quasiparticles eventually collide with each other. Such collisions eventually leads to thermal equilibration in a chaotic quantum state, but the equilibration takes a long time. In a Fermi liquid, this time diverges as

$$\tau_{\rm eq} \sim \frac{\hbar E_F}{(k_B T)^2} \quad , \quad \text{as } T \to 0,$$

where E_F is the Fermi energy.

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  where E_F is the Fermi energy.

- This time is much longer than the 'Planckian time' $\hbar/(k_B T)$, which we will find in systems without quasiparticle excitations.

$$\tau_{\rm eq} \gg \frac{\hbar}{k_B T} \quad , \quad \text{as } T \to 0.$$

1. Random matrix quasiparticle model

q=2, complex SYK

2. Matter without quasiparticles

q=4, complex SYK

3. The Schwarzian theory

4. Connections to black holes with AdS_2 horizons

5. Connections to strange metals

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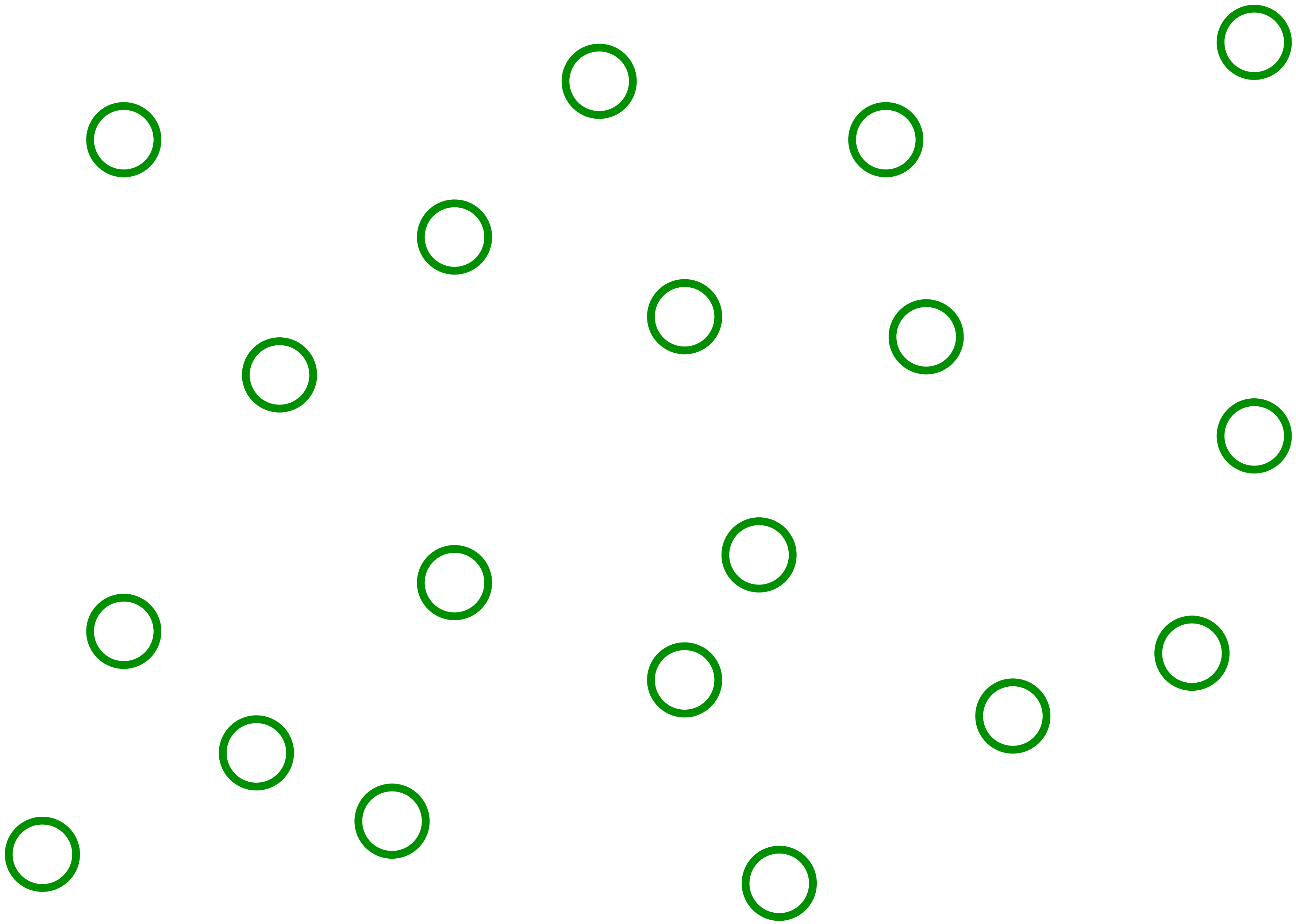
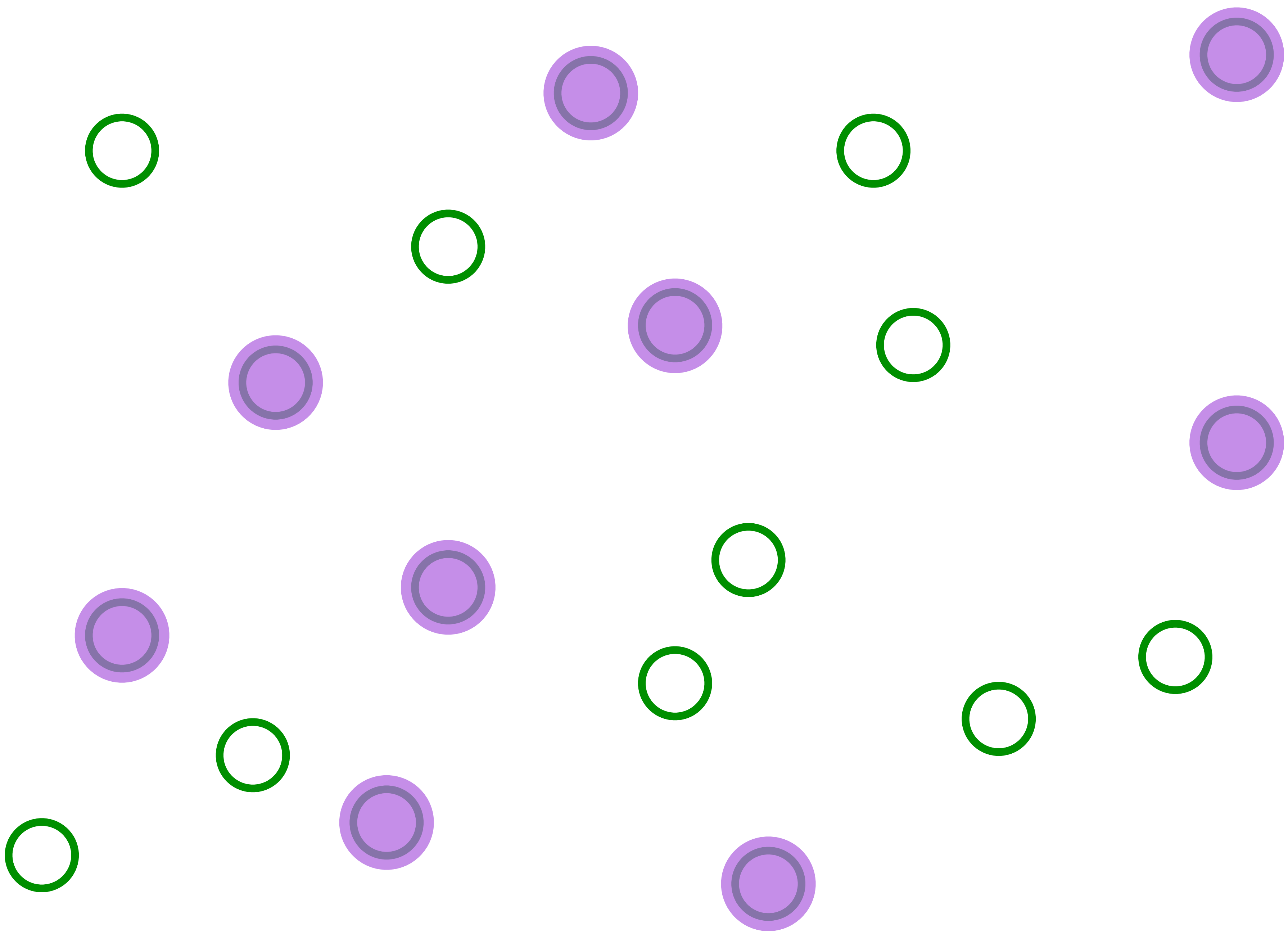
A simple model of a metal with quasiparticles



Pick a set of random positions

A simple model of a metal with quasiparticles



Place electrons randomly on some sites

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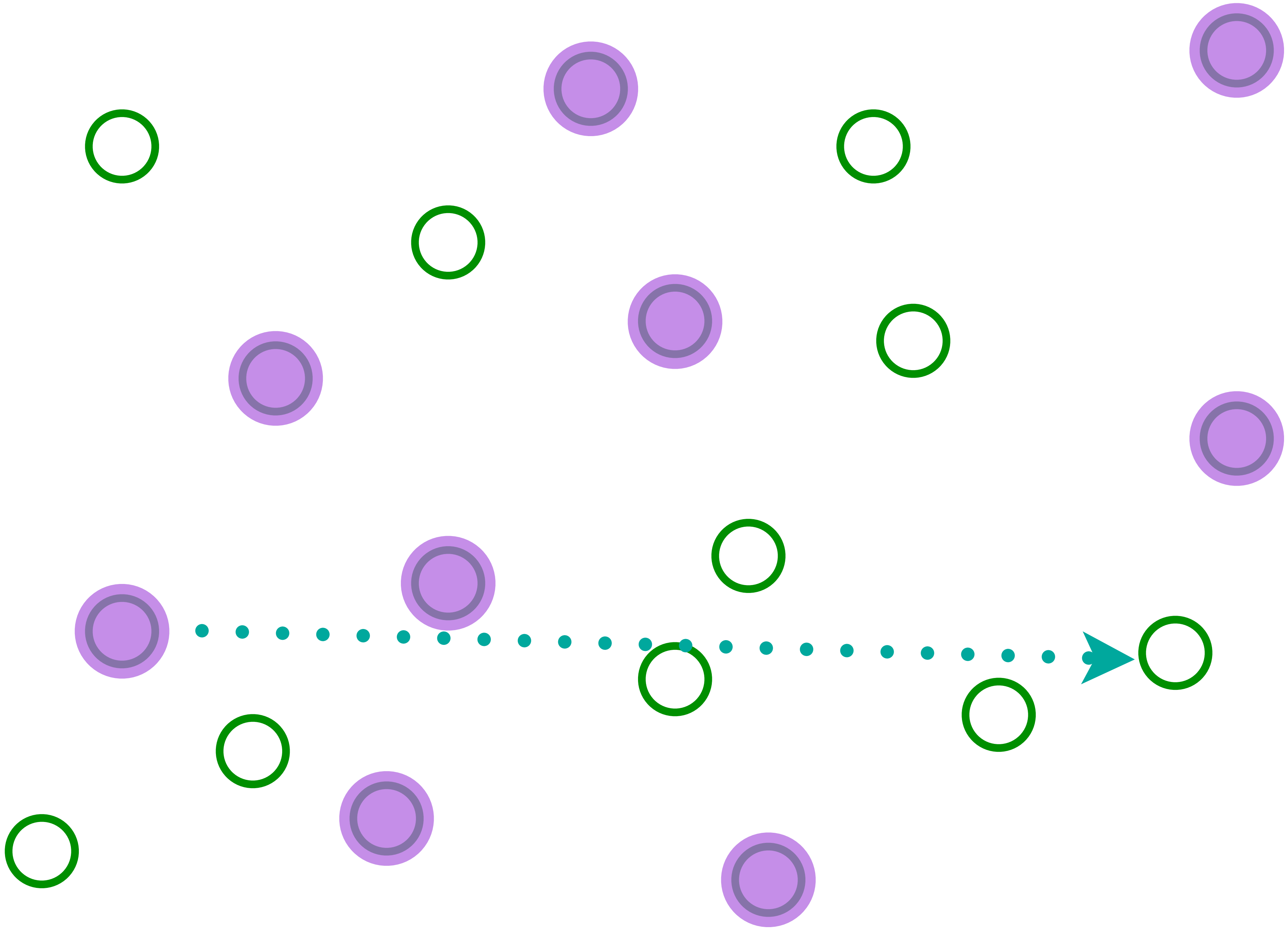

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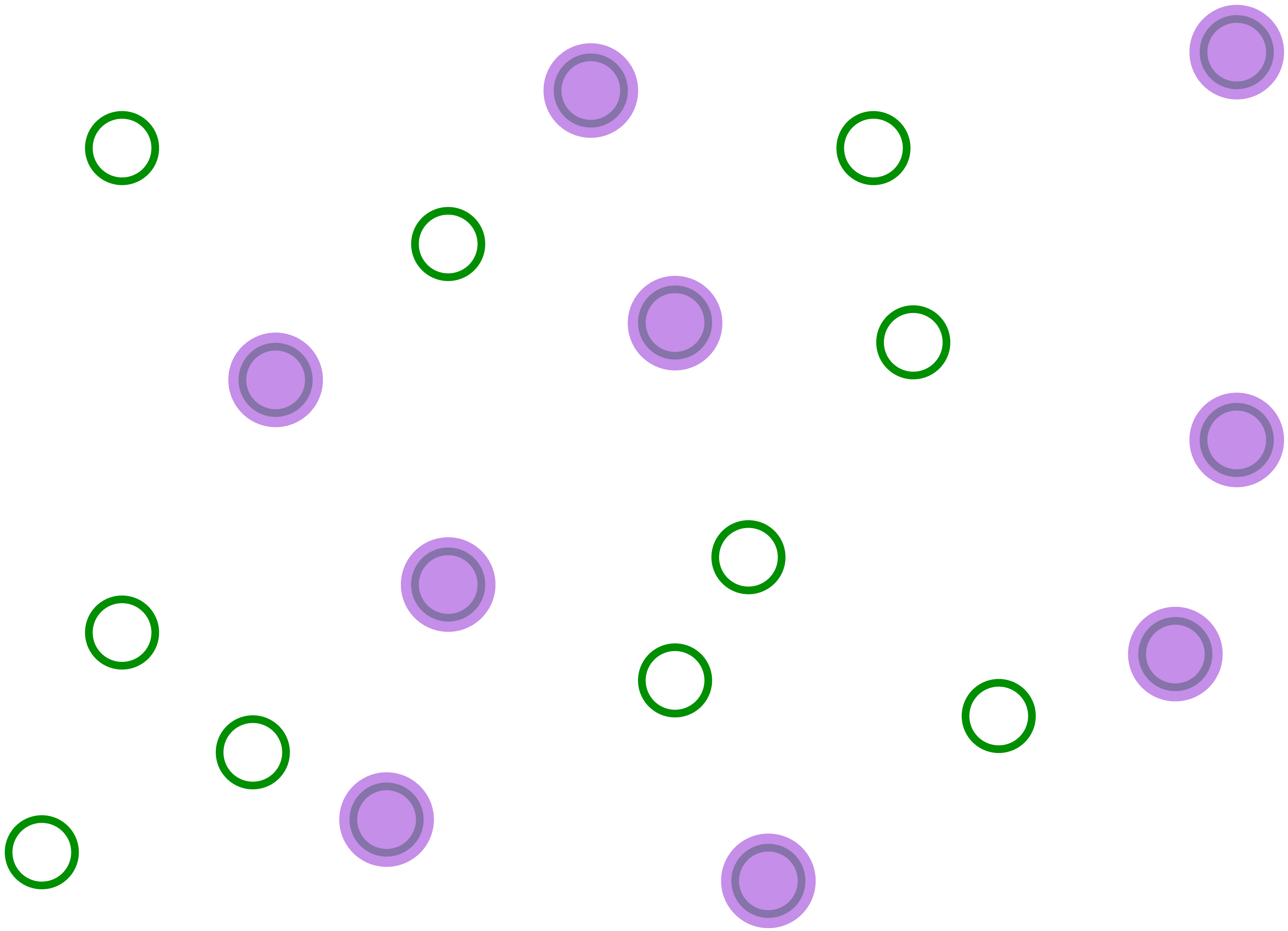

Electrons move one-by-one randomly

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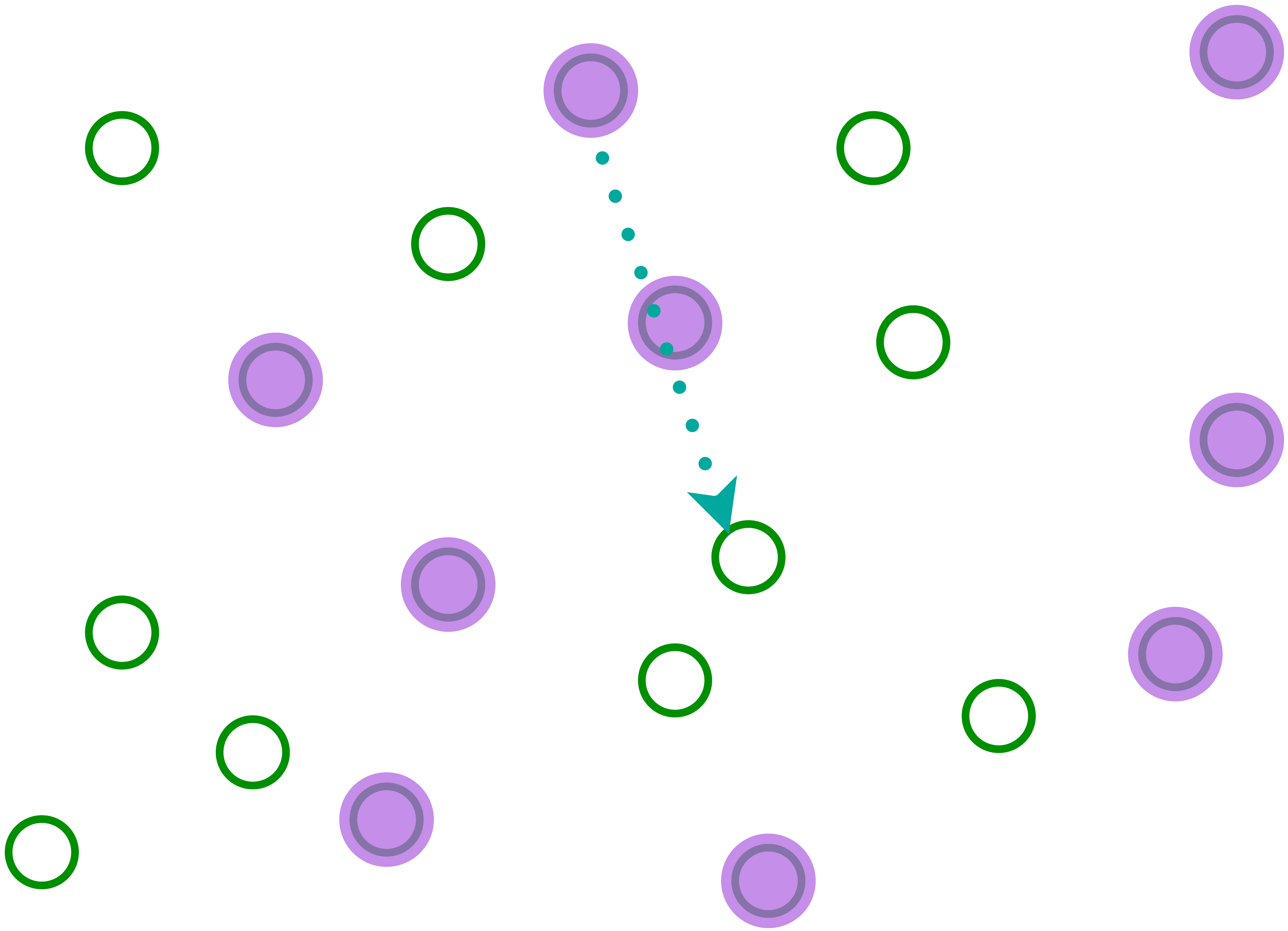

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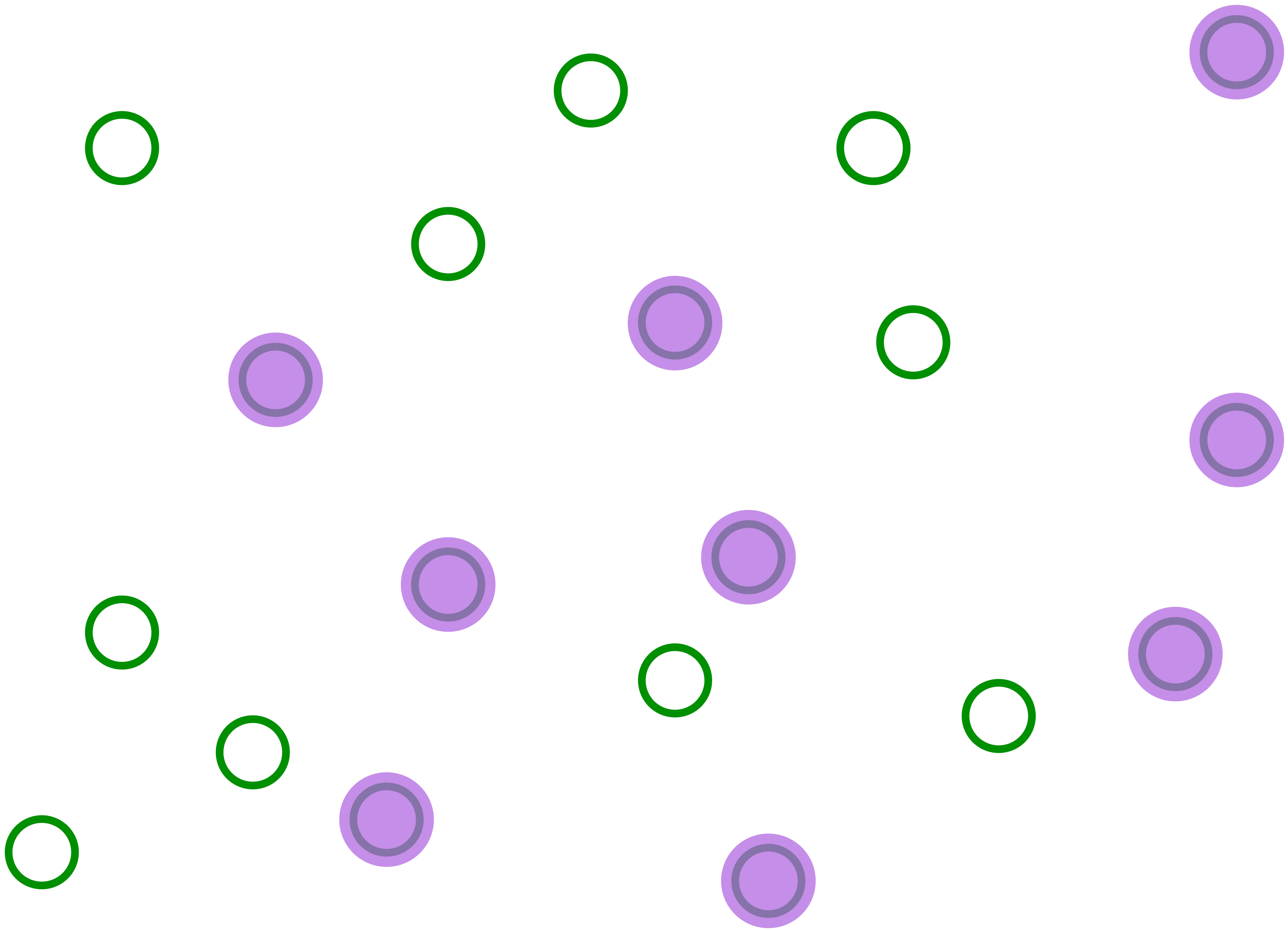

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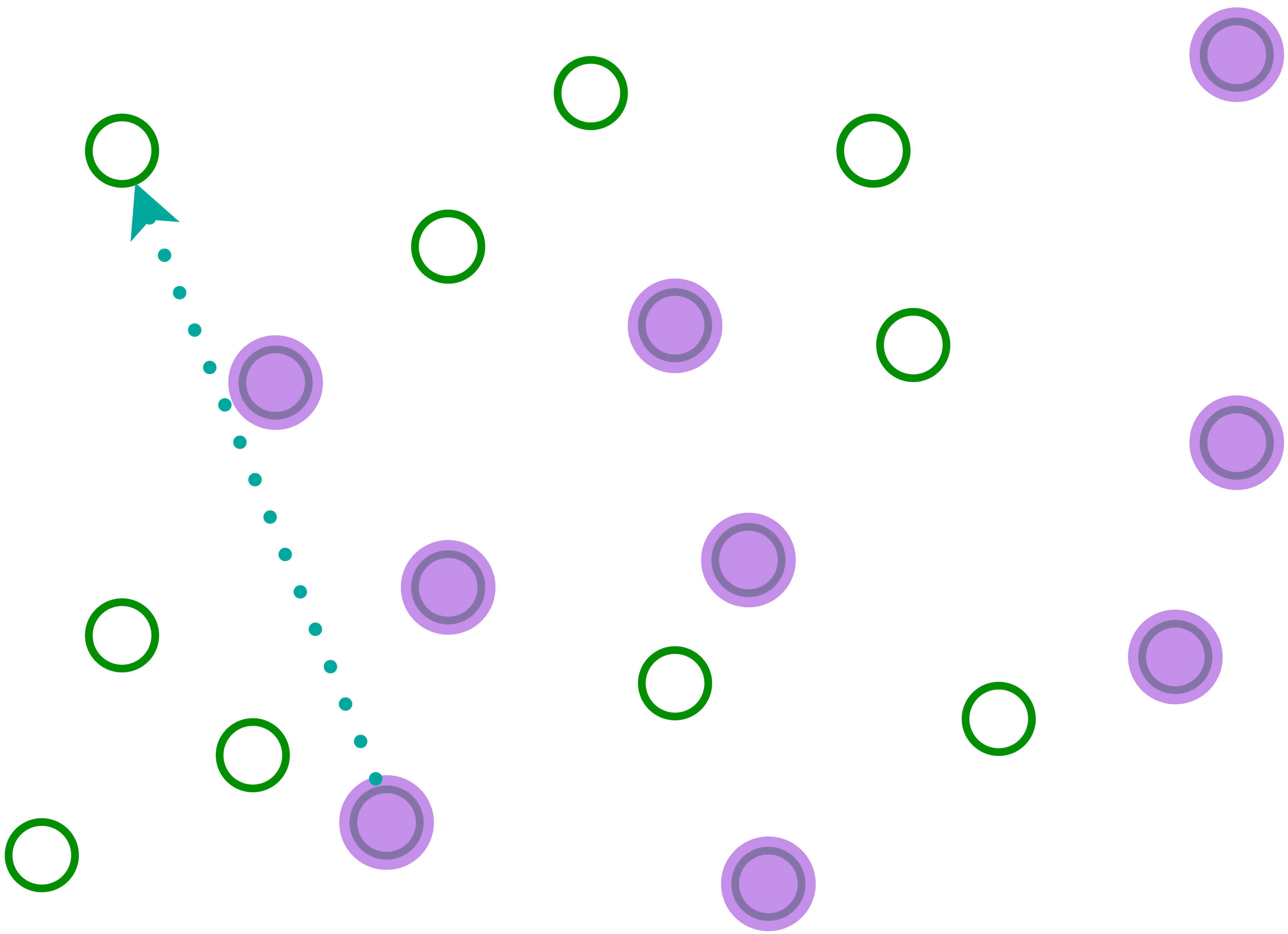

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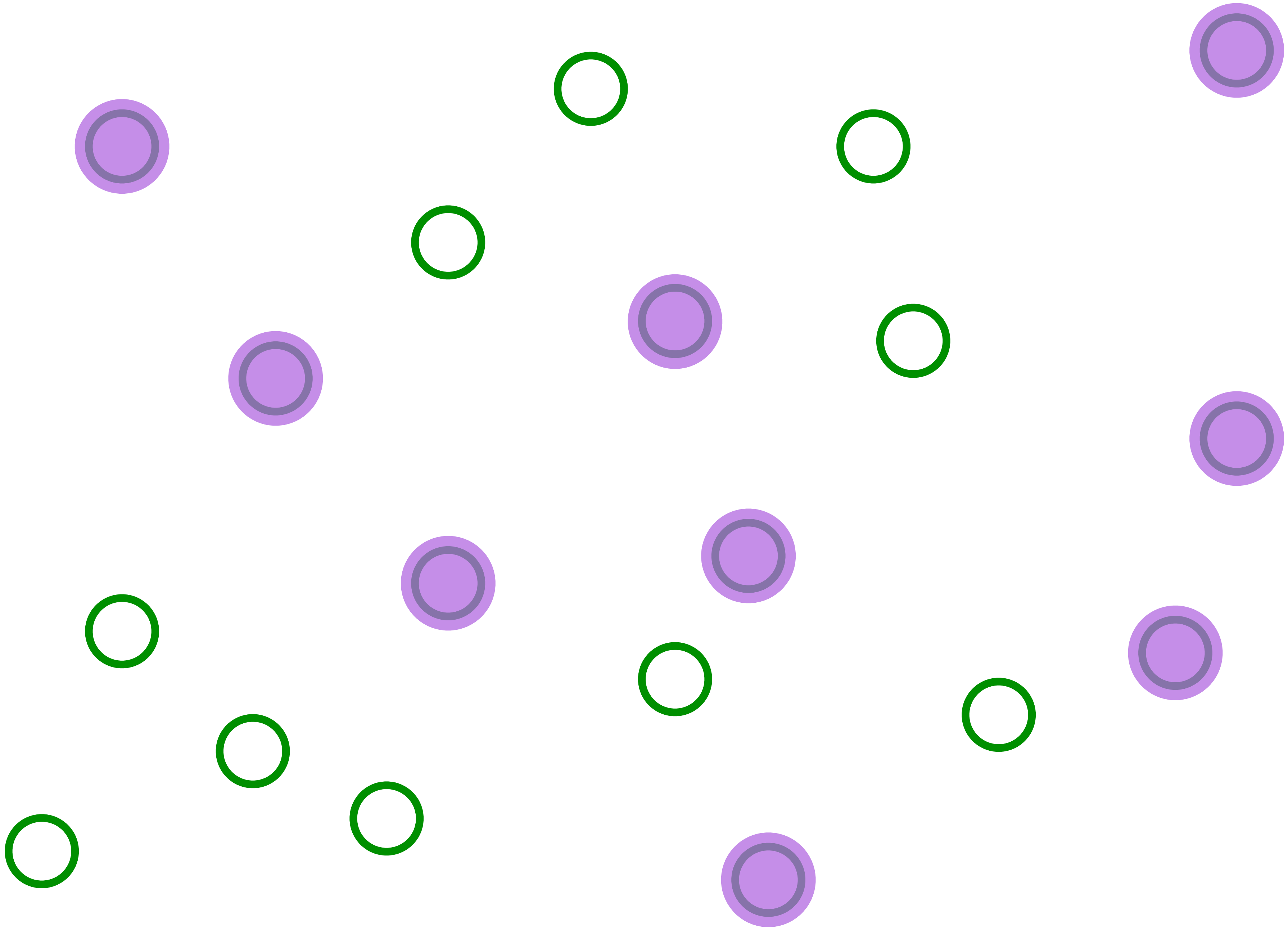

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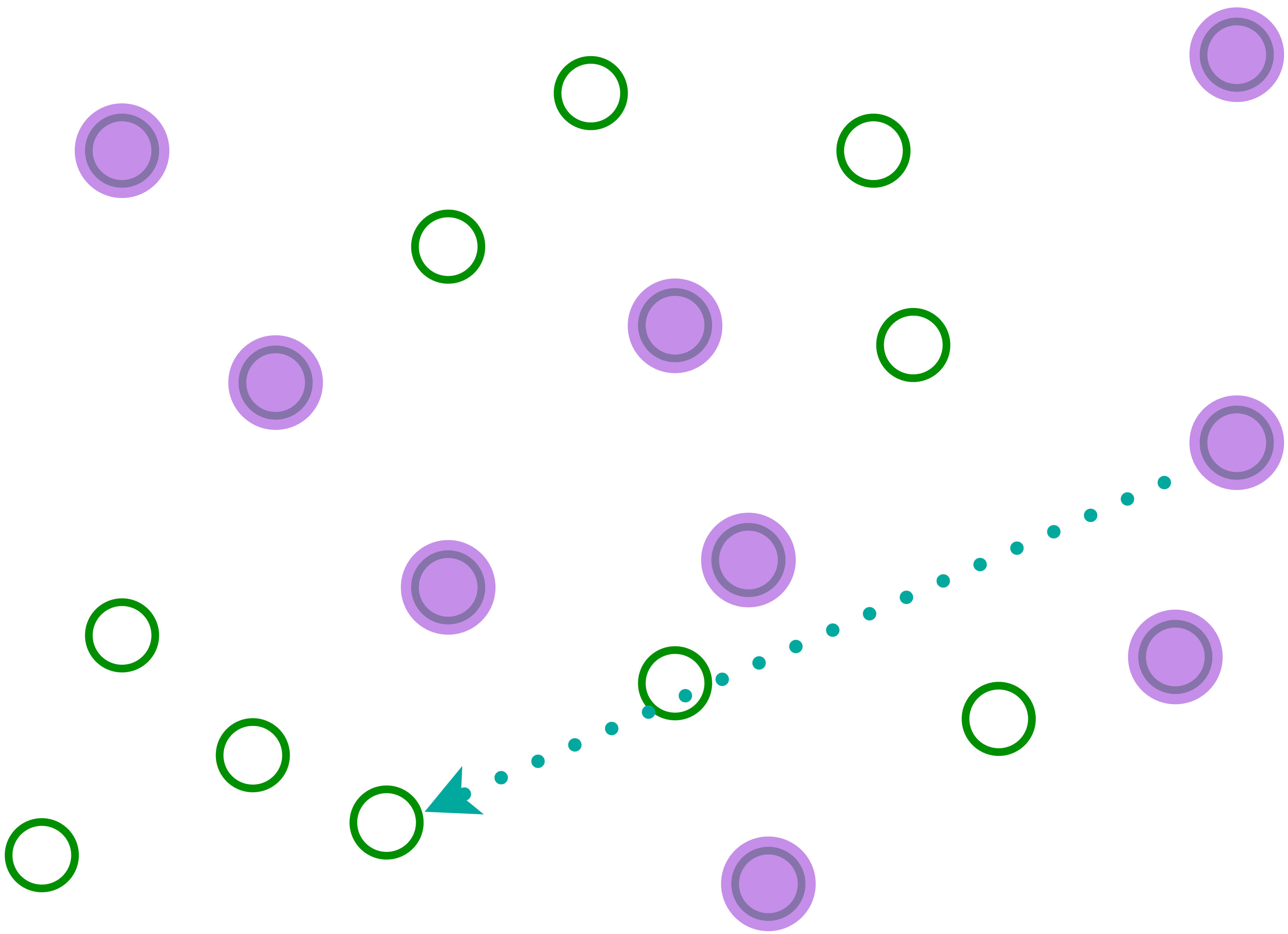

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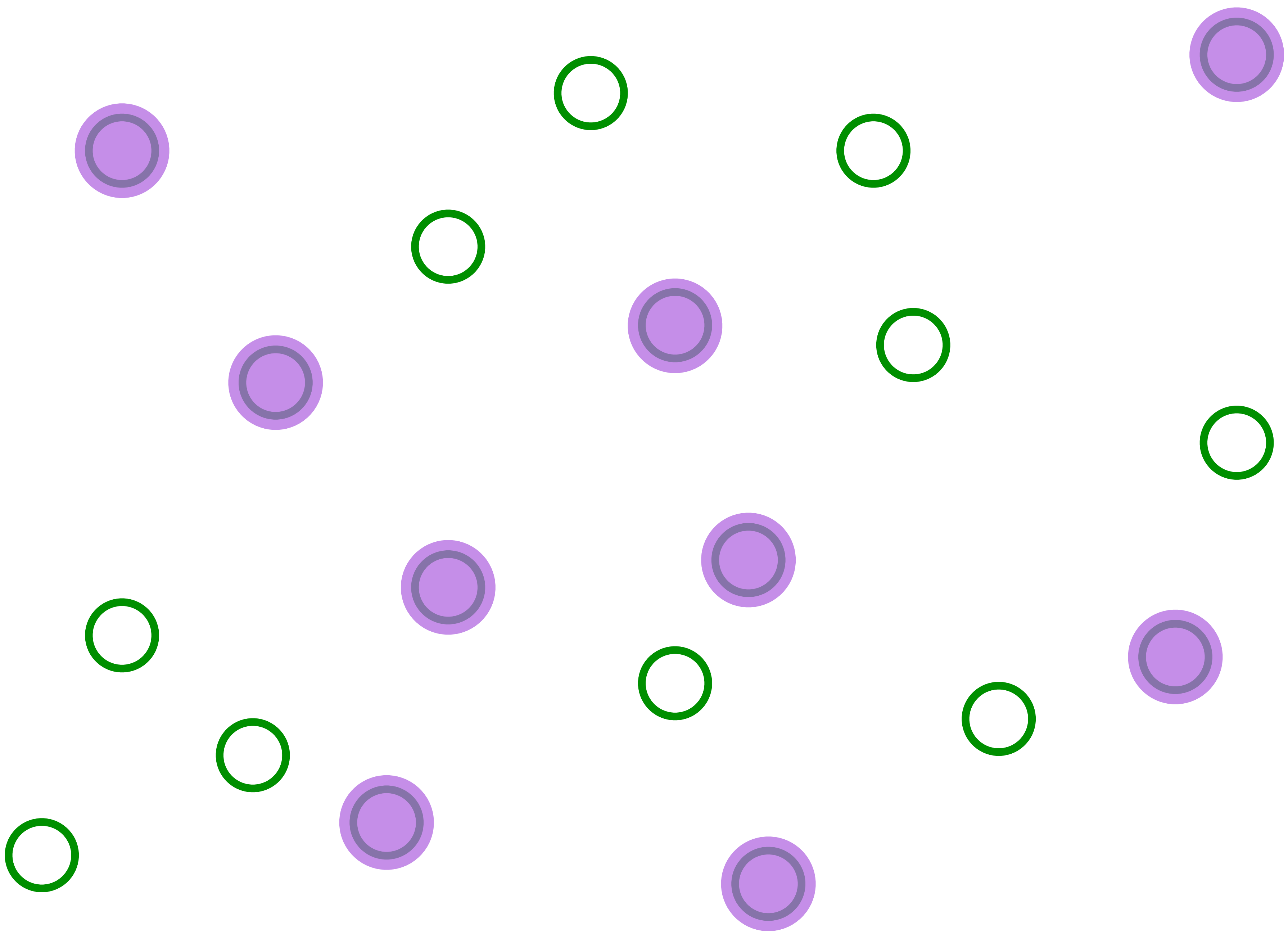


Electrons move one-by-one randomly

A simple model of a metal with quasiparticles

$$H = \frac{1}{(N)^{1/2}} \sum_{i,j=1}^{N} t_{ij} c_i^\dagger c_j - \mu \sum_i c_i^\dagger c_i$$

$$c_i c_j + c_j c_i = 0 \quad , \quad c_i c_j^\dagger + c_j^\dagger c_i = \delta_{ij}$$

$$\frac{1}{N} \sum_i c_i^\dagger c_i = \mathcal{Q}$$

t_{ij} are independent random variables with $\overline{t_{ij}} = 0$ and $\overline{|t_{ij}|^2} = t^2$

Fermions occupying the eigenstates of a
N x *N* random matrix

A simple model of a metal with quasiparticles

Feynman graph expansion in $t_{ij..}$, and graph-by-graph average, yields exact equations in the large N limit:

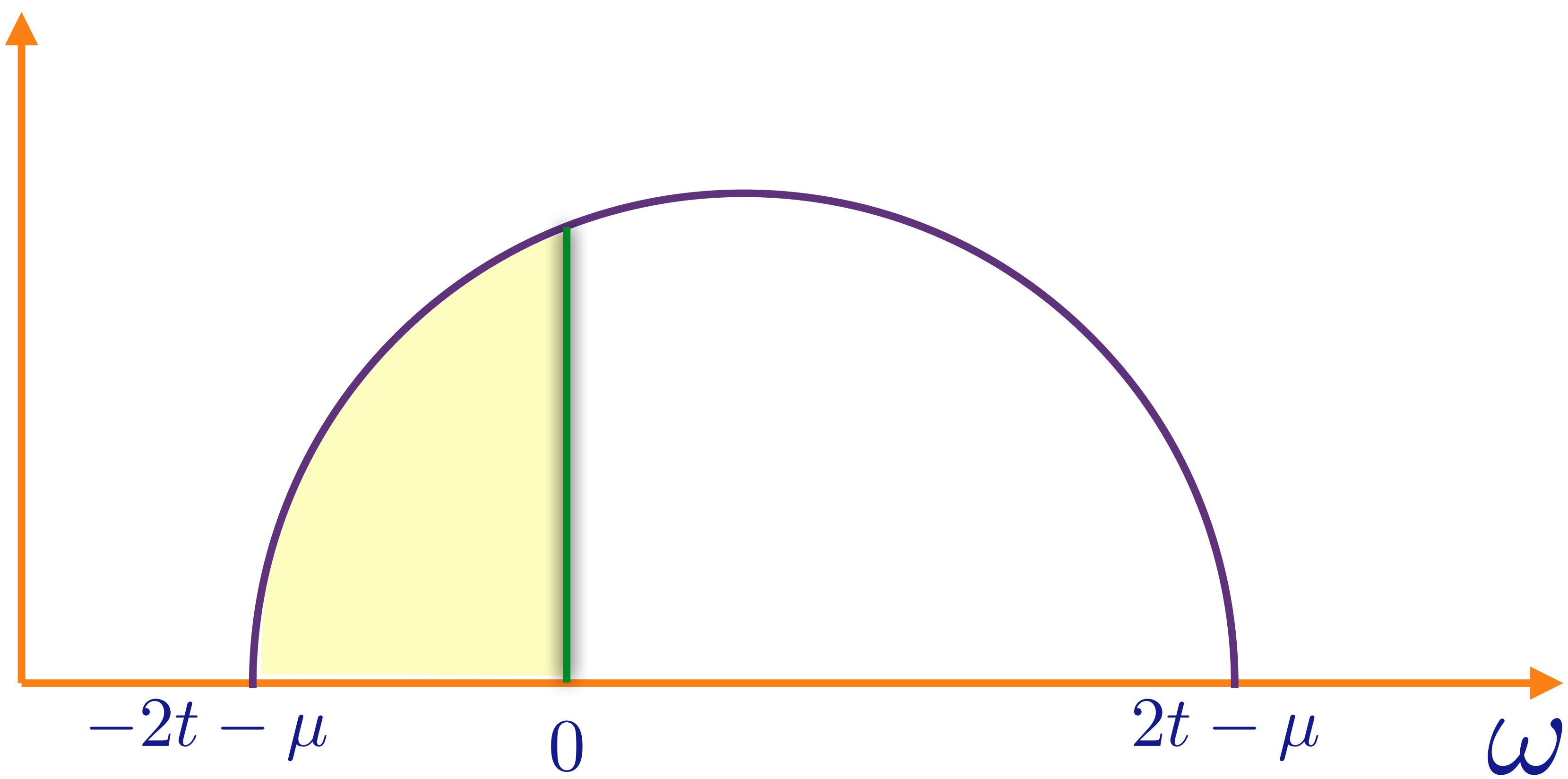
$$G(\tau) \equiv -T_\tau \left\langle c_i(\tau) c_i^\dagger(0) \right\rangle$$

$$G(i\omega) = \frac{1}{i\omega + \mu - \Sigma(i\omega)} \quad , \quad \Sigma(\tau) = t^2 G(\tau)$$

$$G(\tau = 0^-) = \mathcal{Q}.$$

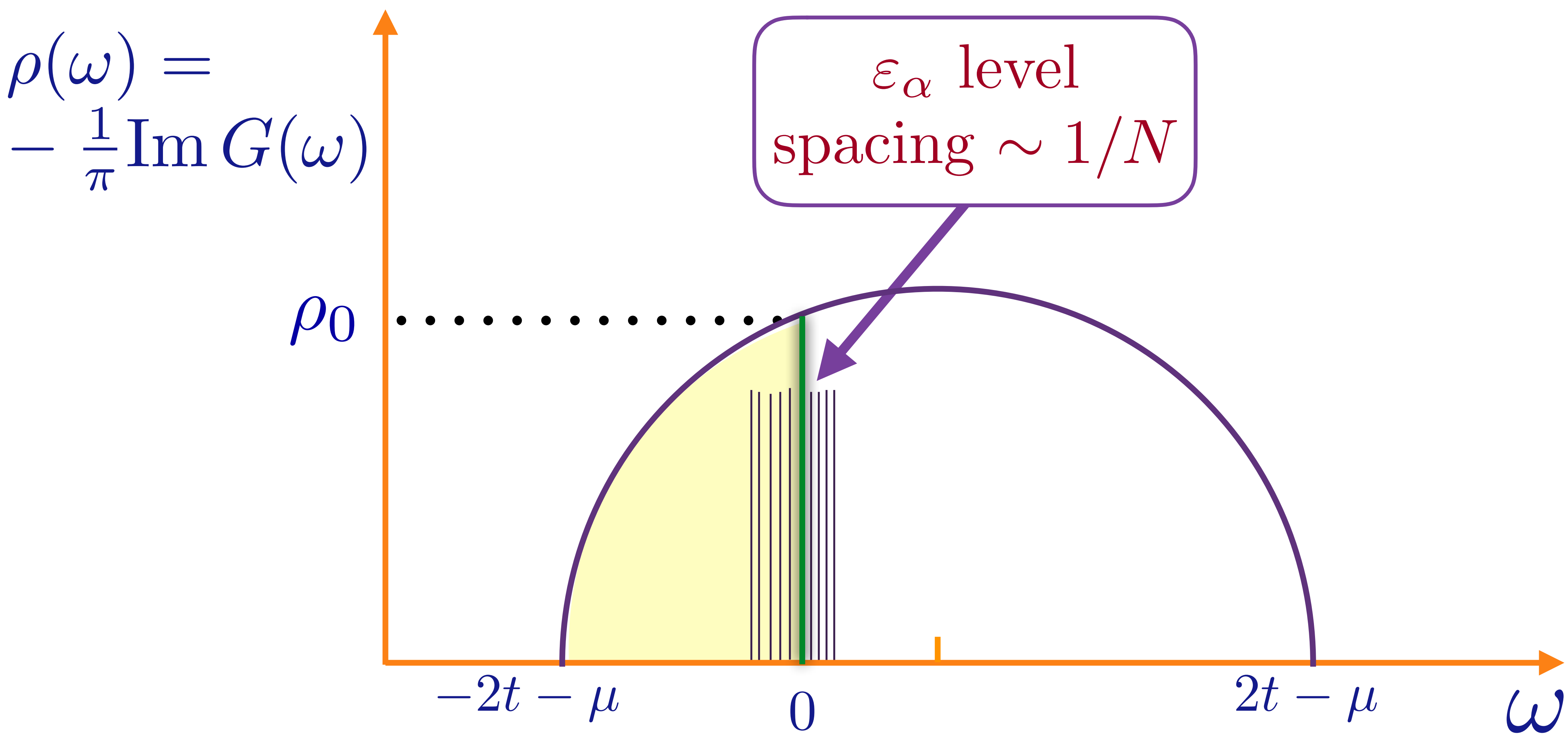
$G(\omega)$ can be determined by solving a quadratic equation.

$\rho(\omega) =$
$-\frac{1}{\pi} \mathrm{Im}\, G(\omega)$



A simple model of a metal with quasiparticles

Let ε_α be the eigenvalues of the matrix $t_{ij}/\sqrt{N}$. The fermions will occupy the lowest NQ eigenvalues, upto the Fermi energy E_F. The single-particle density of states is
$\rho(\omega) = (1/N)\sum_\alpha \delta(\omega - \varepsilon_\alpha)$, and $\rho_0 \equiv \rho(\omega = 0)$.



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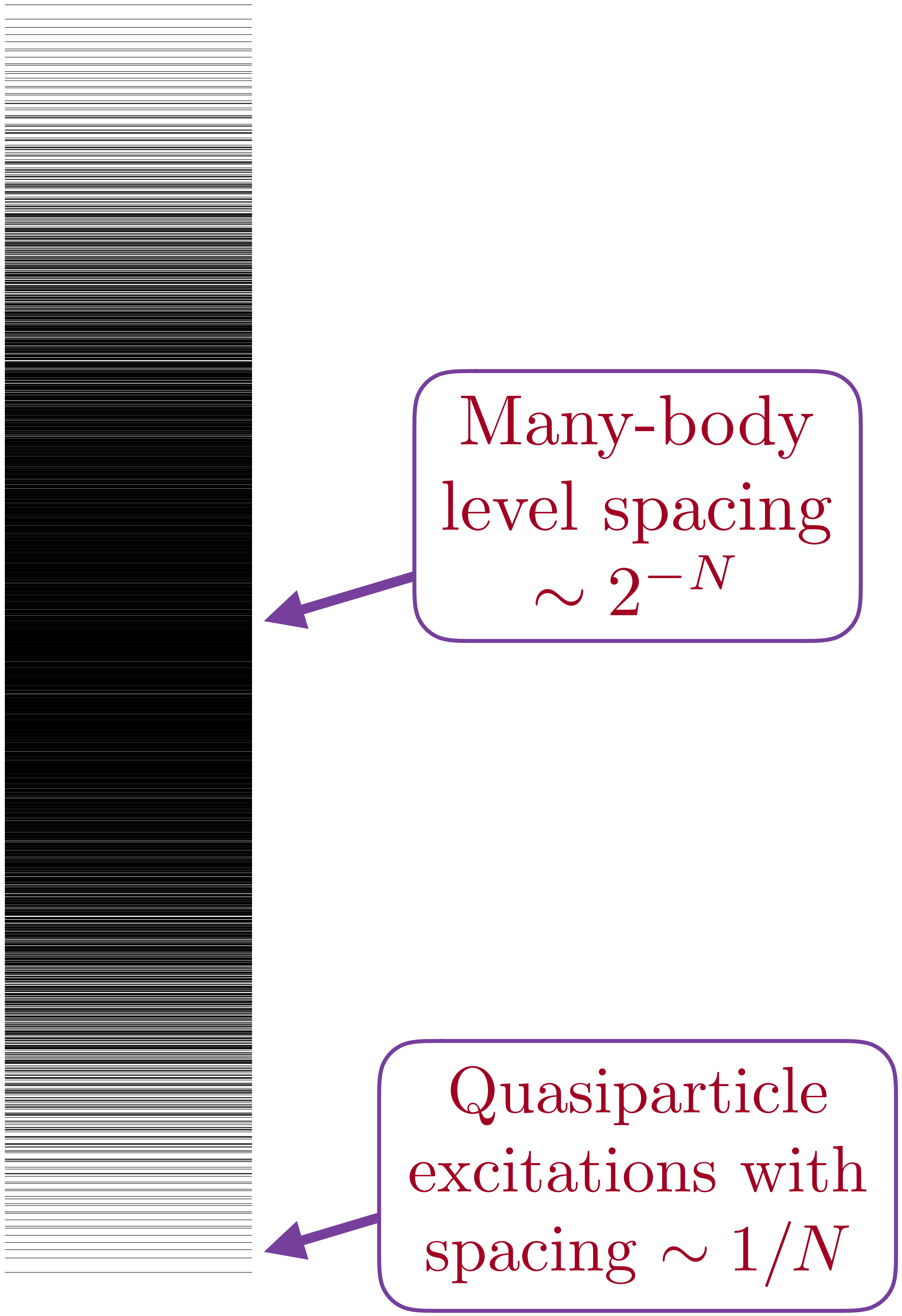

There are 2^N many body levels with energy

$$E = \sum_{\alpha=1}^{N} n_\alpha \varepsilon_\alpha ,$$

where $n_\alpha = 0, 1$. Shown are all values of E for a single cluster of size $N = 12$. The ε_α have a level spacing $\sim 1/N$.

A simple model of a metal with quasiparticles

The grand potential $\Omega(T)$ at low T is (from the Sommerfeld expansion)

$$\Omega(T) - E_0 = N\left(-\frac{\pi^2}{6}\rho_0 T^2 + \mathcal{O}(T^4)\right) + \ldots$$

where $\rho_0 \equiv \rho(0)$ is the *single* particle density of states at the Fermi level.
We can also define the *many* body density of states, $D(E)$, via

$$Z = e^{-\Omega(T)/T} = \int_{-\infty}^{\infty} D(E) e^{-E/T}$$

The inversion from $\Omega(T)$ to $D(E)$ has to performed with care (it need not commute with the $1/N$ expansion), and we obtain

$$\boxed{D(E) \sim \exp\left(\pi\sqrt{\frac{2N\rho_0(E-E_0)}{3}}\right) \quad , \quad E > E_0 \ , \ \frac{1}{N} \ll \rho_0(E-E_0) \ll N}$$

and $D(E) = 0$ for $E < E_0$. This is related to the asymptotic growth of the partitions of an integer, $p(n) \sim \exp(\pi\sqrt{2n/3})$. Near the lower bound, there are large sample-to-sample fluctuations due to variations in the lowest quasiparticle energies.

A simple model of a metal with quasiparticles

Now add weak interactions

$$H = \frac{1}{(N)^{1/2}} \sum_{i,j=1}^{N} t_{ij} c_i^\dagger c_j - \mu \sum_i c_i^\dagger c_i + \frac{1}{(2N)^{3/2}} \sum_{i,j,k,\ell=1}^{N} U_{ij;k\ell}\, c_i^\dagger c_j^\dagger c_k c_\ell$$

$U_{ij;k\ell}$ are independent random variables with $\overline{U_{ij;k\ell}} = 0$ and $\overline{|U_{ij;k\ell}|^2} = U^2$. We compute the lifetime of a quasiparticle, τ_α, in an exact eigenstate $\psi_\alpha(i)$ of the free particle Hamitonian with energy ε_α. By Fermi's Golden rule, for ε_α at the Fermi energy

$$\begin{aligned} \frac{1}{\tau_\alpha} &= \pi U^2 \rho_0^2 \int d\varepsilon_\beta d\varepsilon_\gamma d\varepsilon_\delta f(\varepsilon_\beta)(1 - f(\varepsilon_\gamma))(1 - f(\varepsilon_\delta))\delta(\varepsilon_\alpha + \varepsilon_\beta - \varepsilon_\gamma - \varepsilon_\delta) \\ &= \frac{\pi^3 U^2 \rho_0^2}{4} T^2 \end{aligned}$$

where ρ_0 is the density of states at the Fermi energy, and $f(\epsilon) = 1/(e^{\epsilon/T} + 1)$ is the Fermi function.

Fermi liquid state: Two-body interactions lead to a scattering time of quasiparticle excitations from in (random) single-particle eigenstates which diverges as $\sim T^{-2}$ at the Fermi level.

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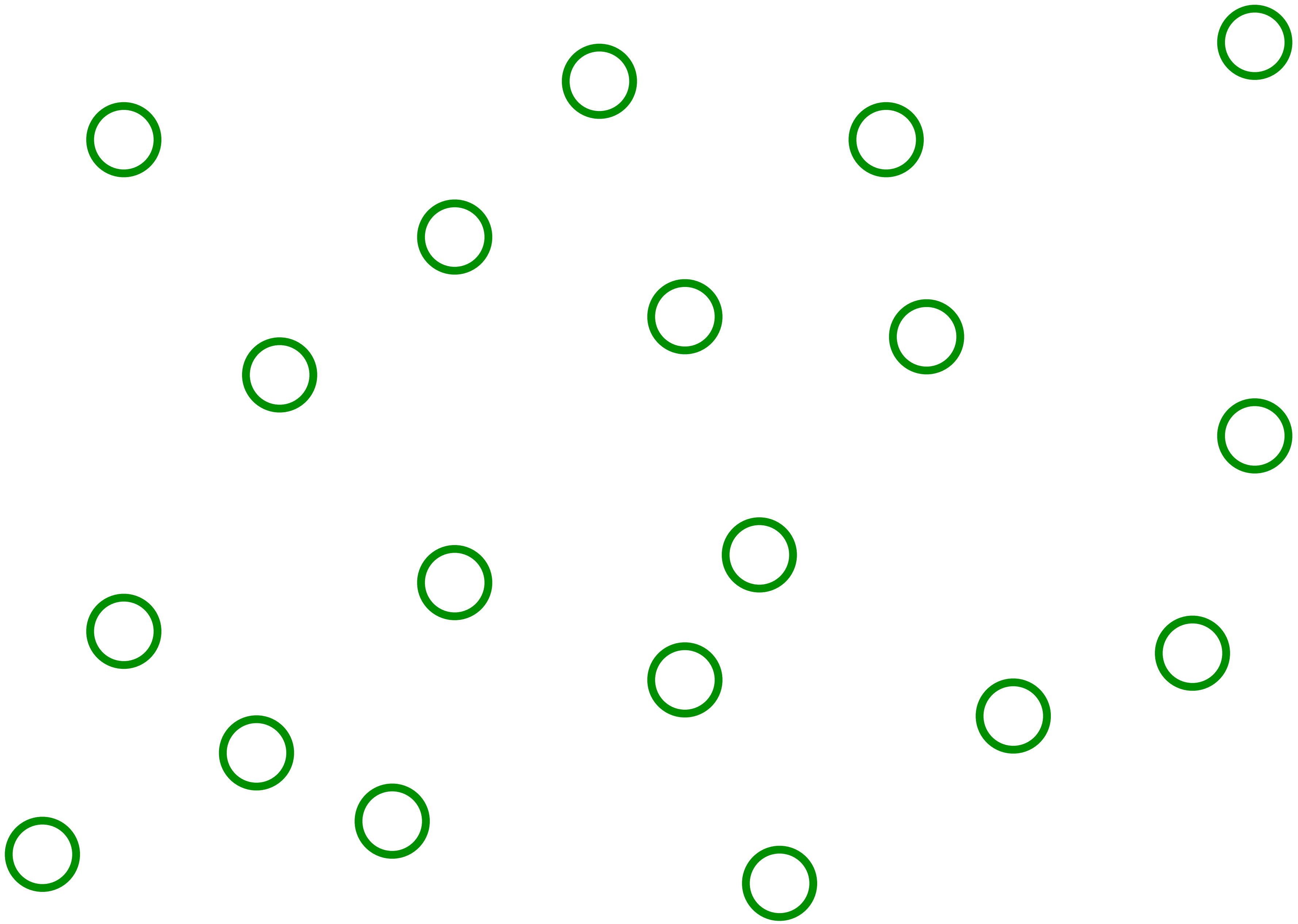
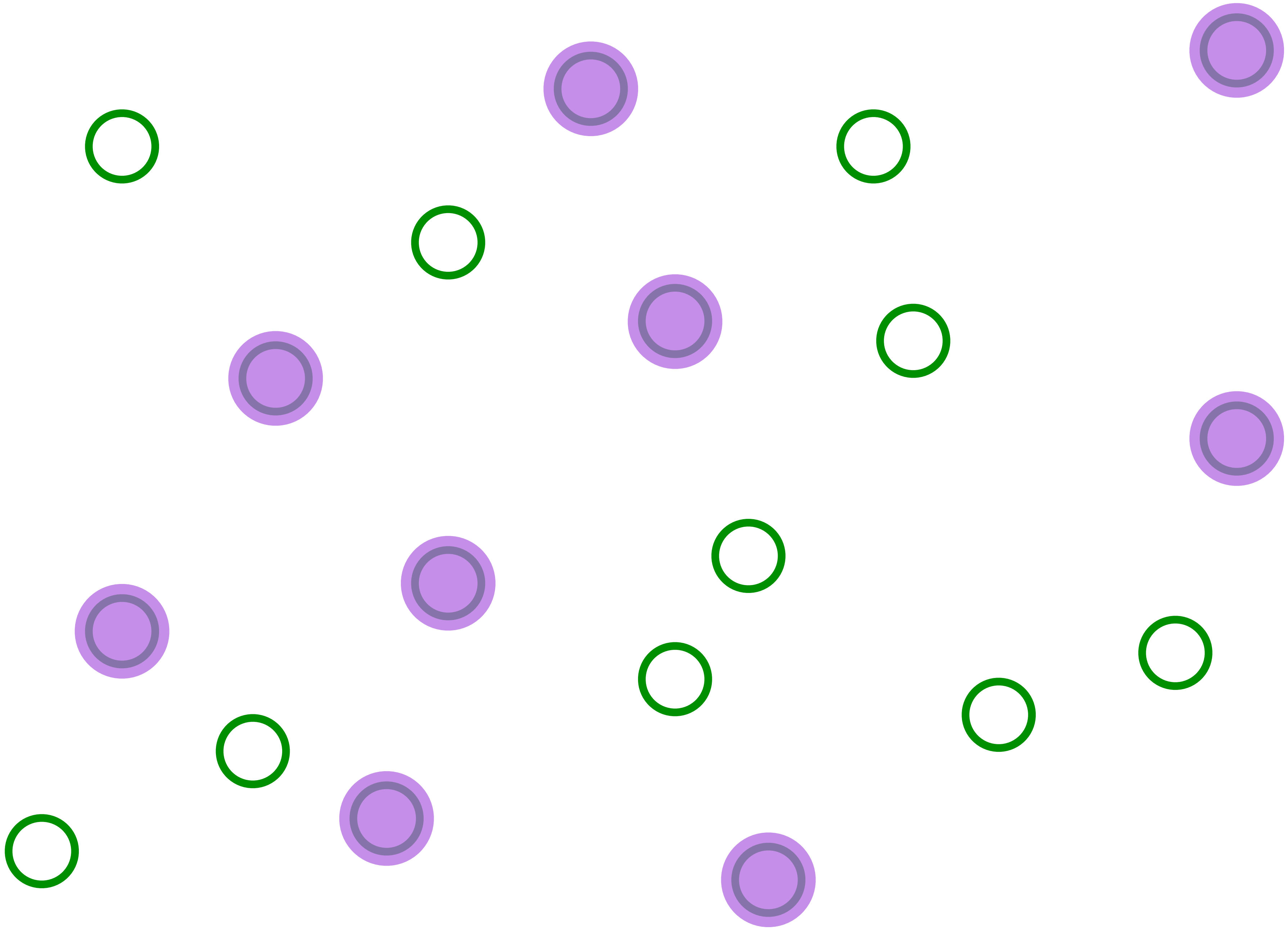
The Sachdev-Ye-Kitaev (SYK) model



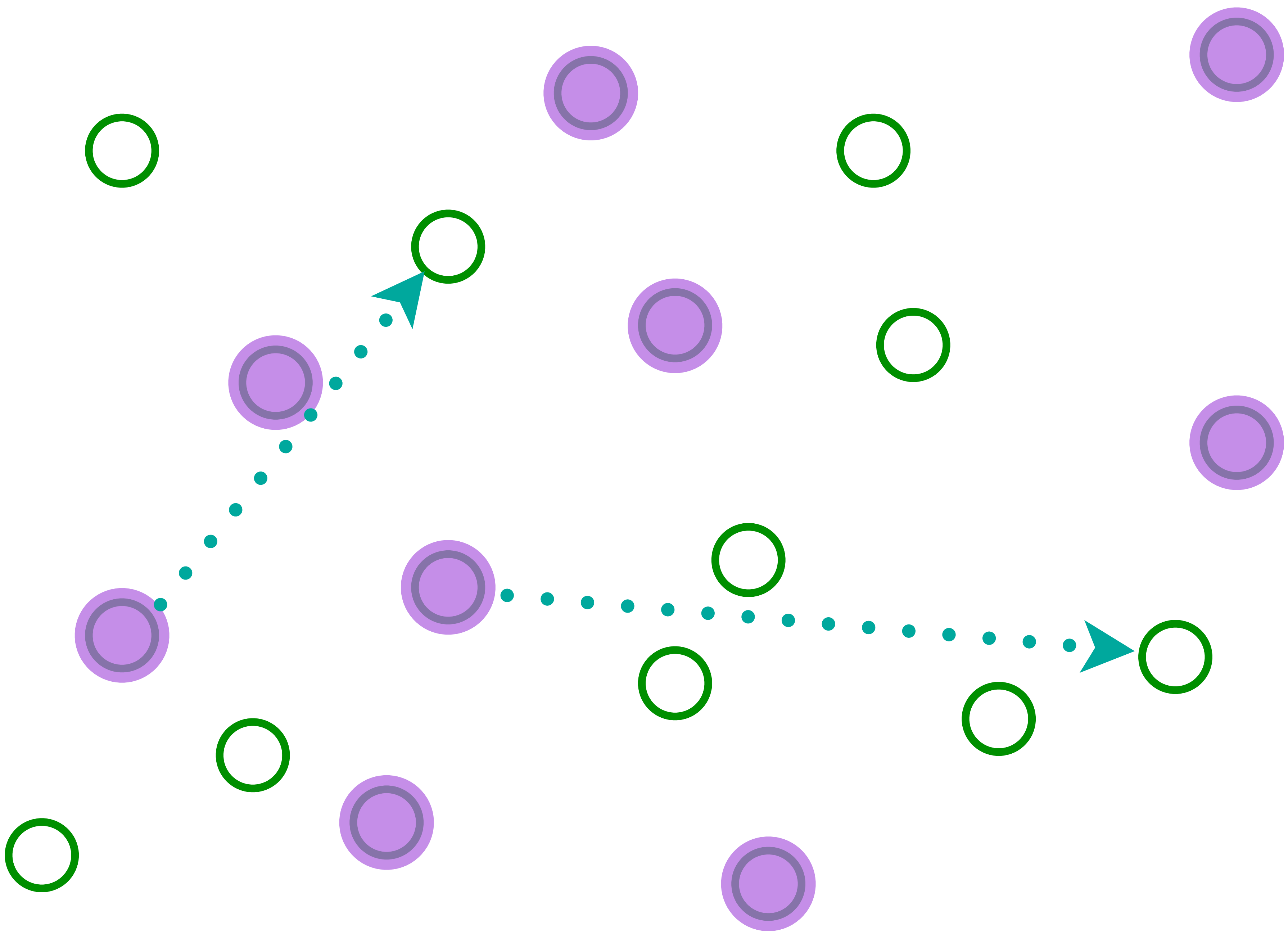
Pick a set of random positions

The SYK model



Place electrons randomly on some sites

The SYK model



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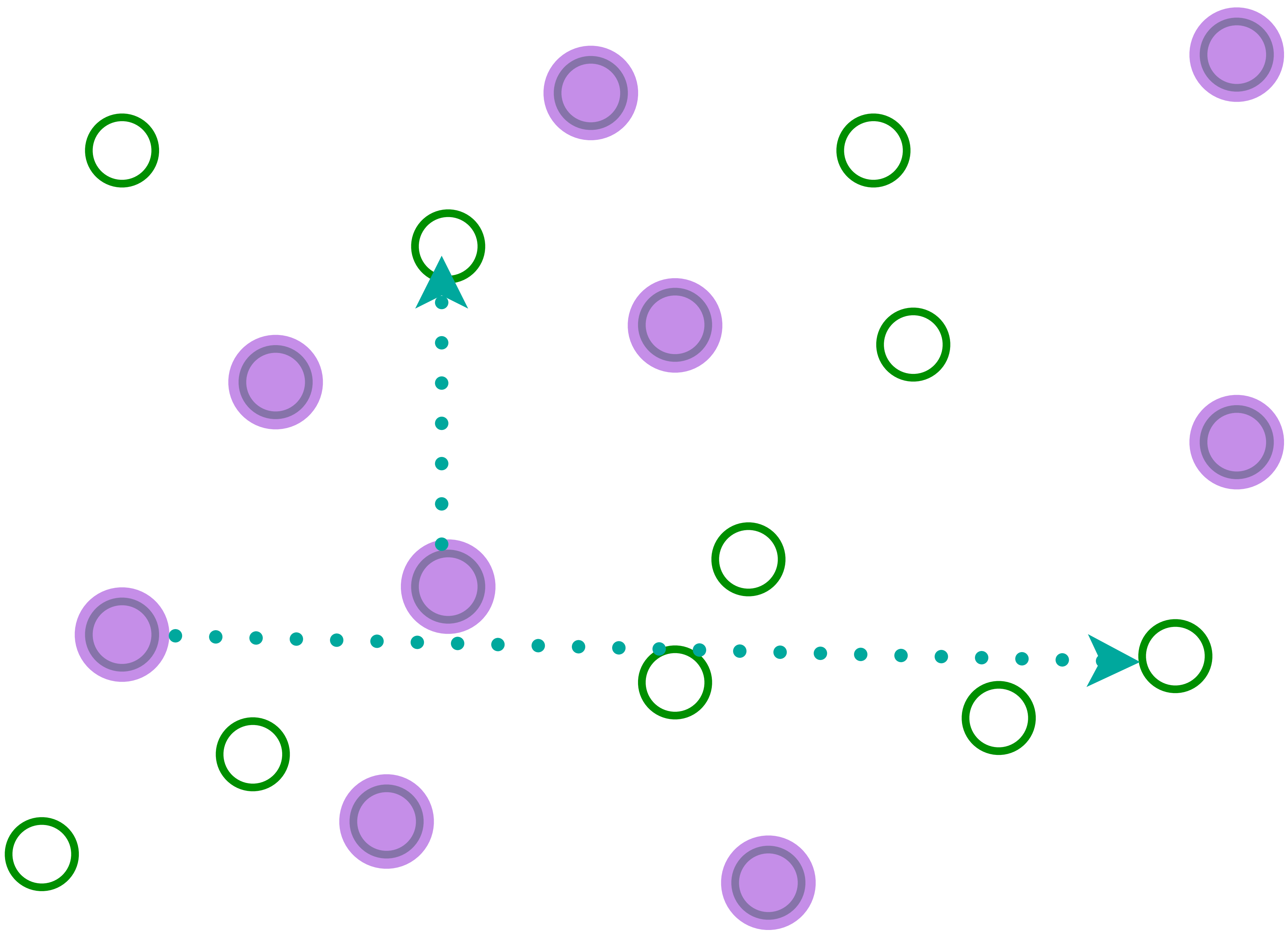

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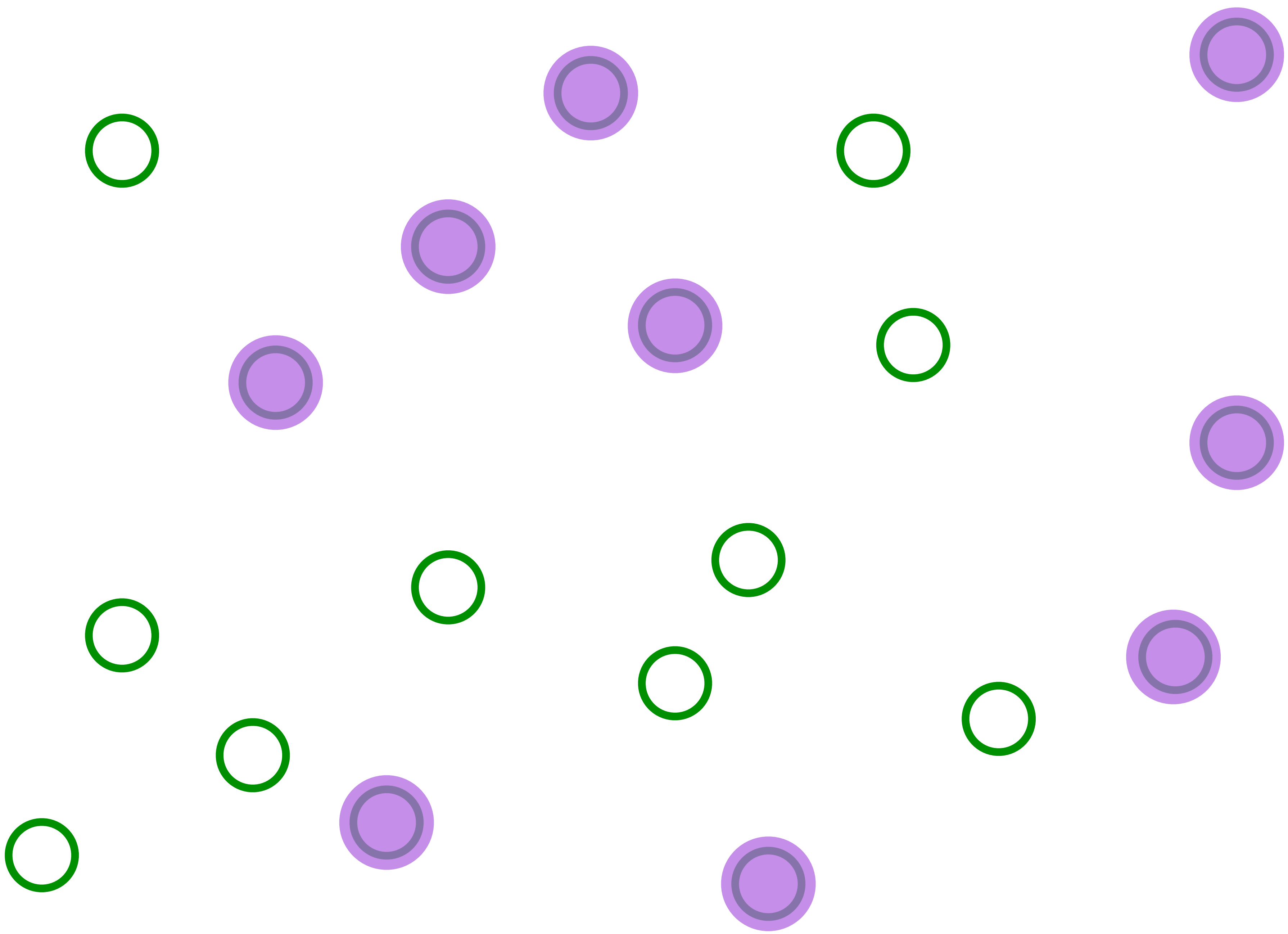

Entangle electrons pairwise randomly

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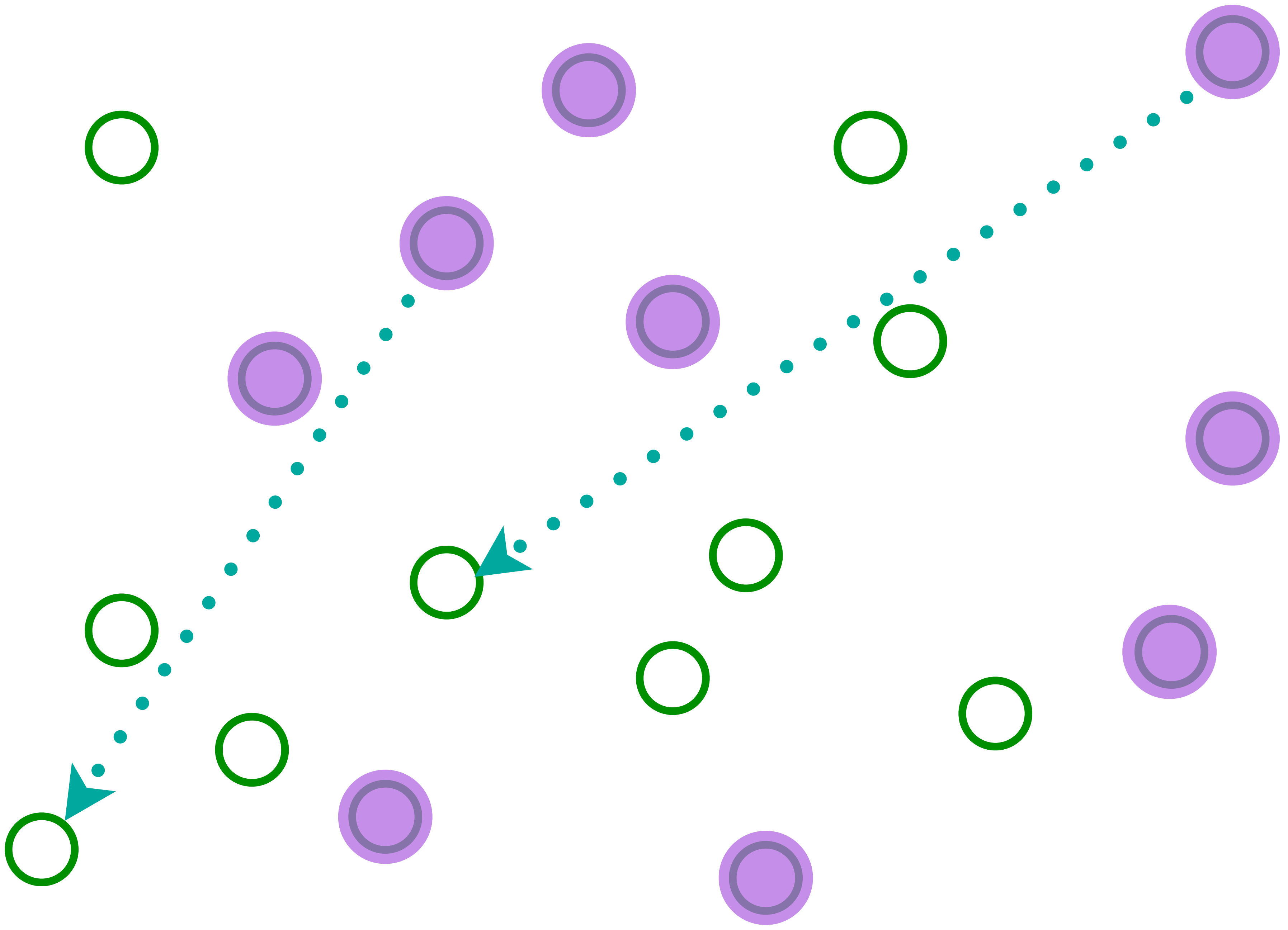

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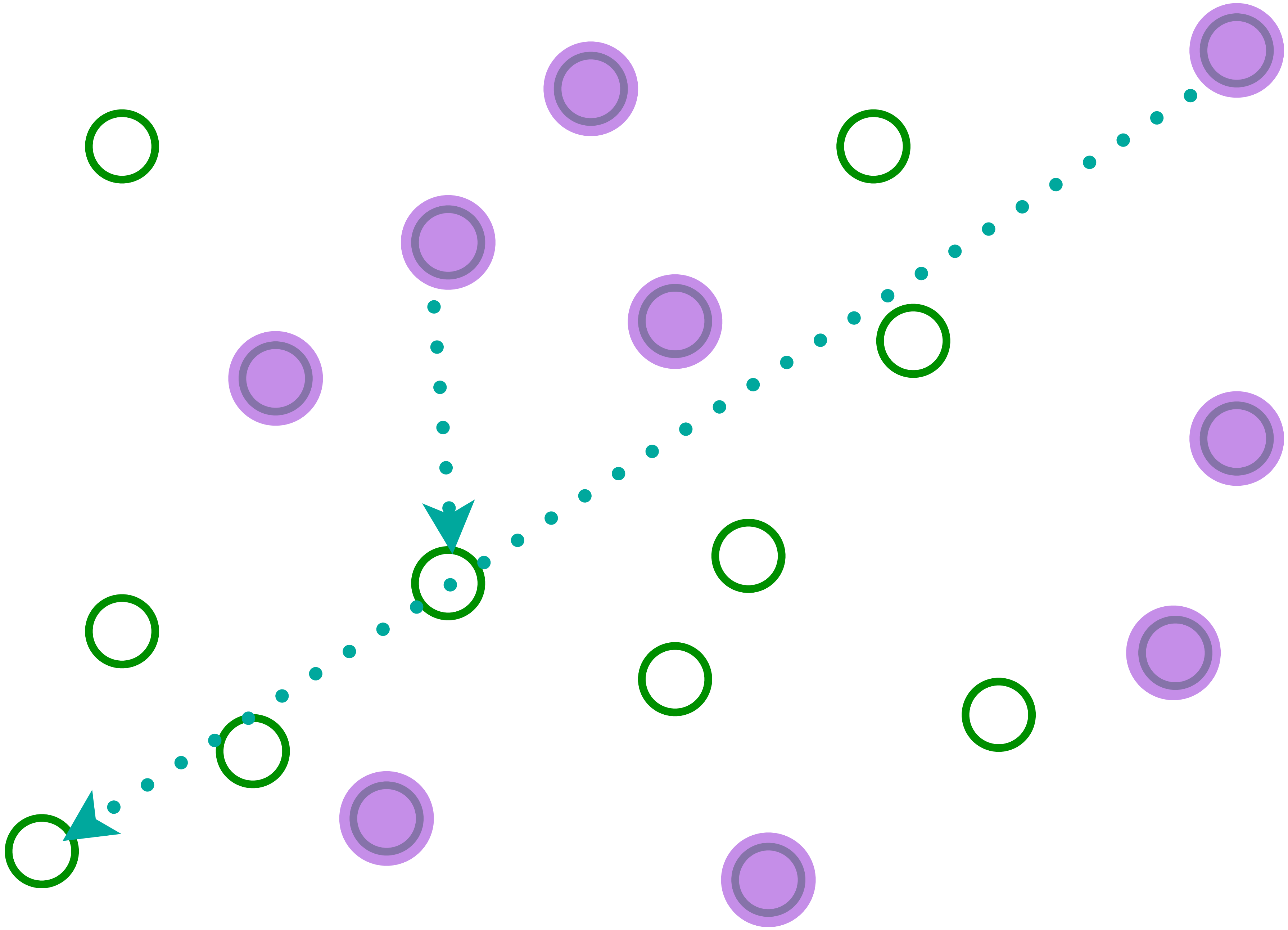

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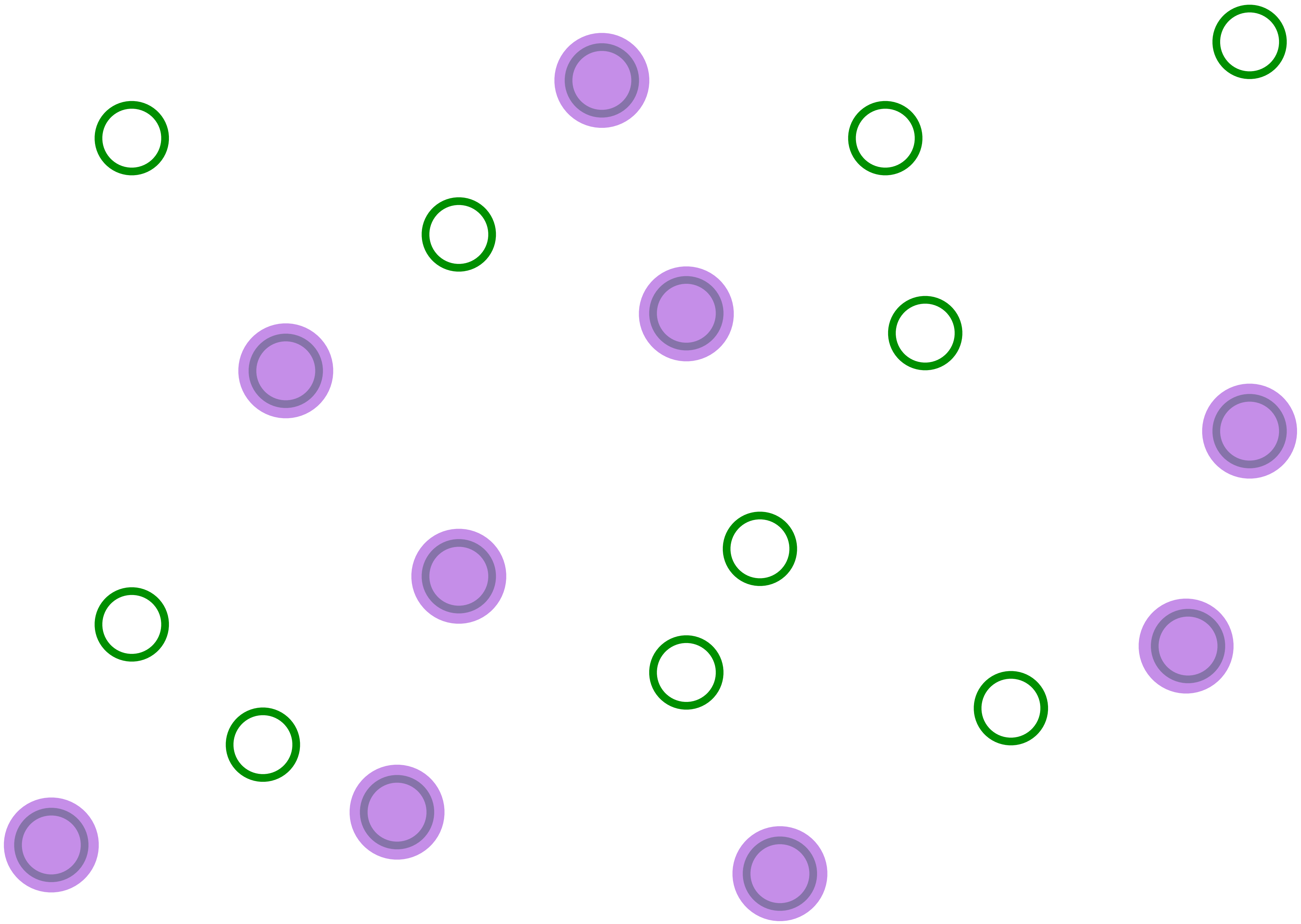

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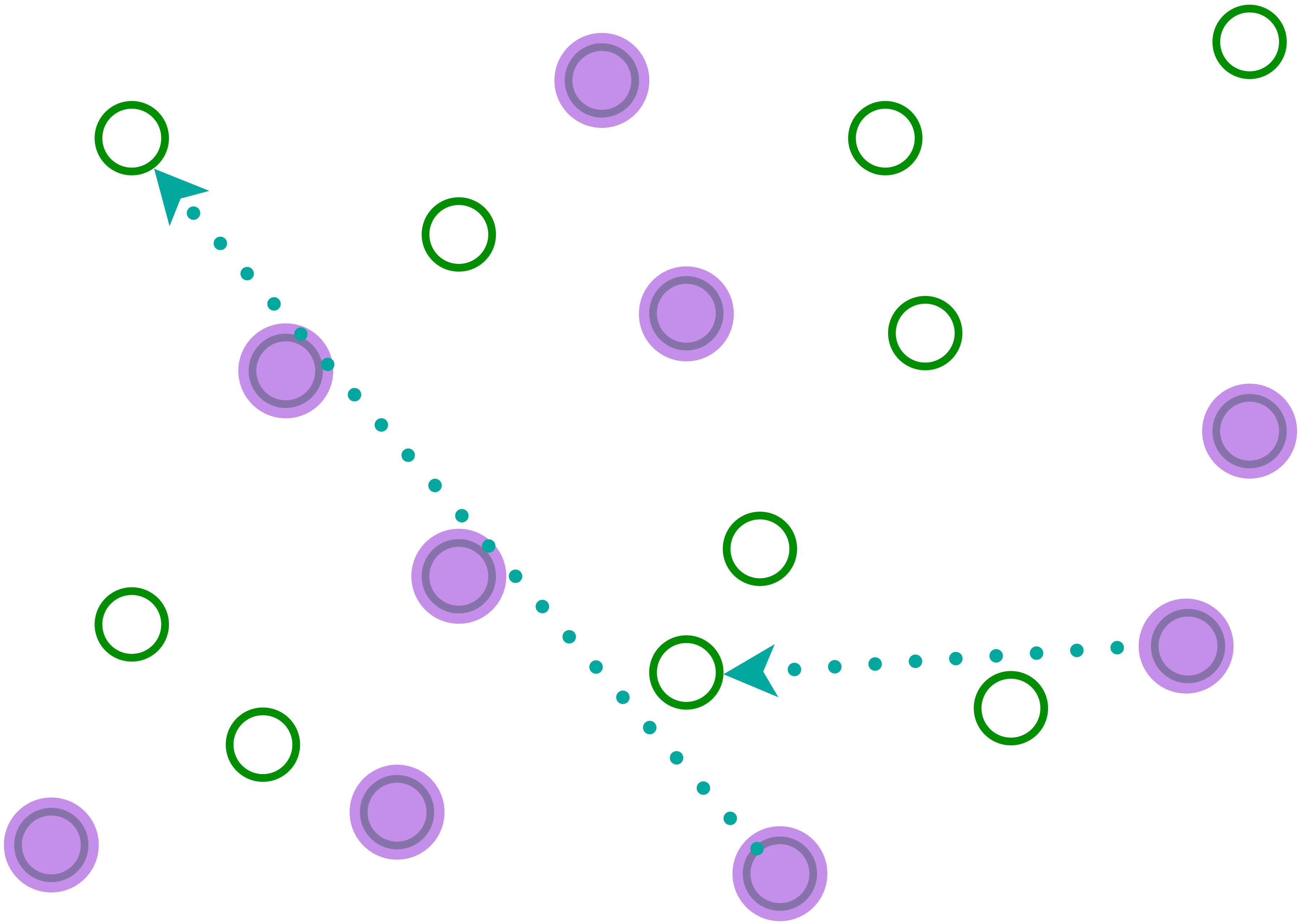

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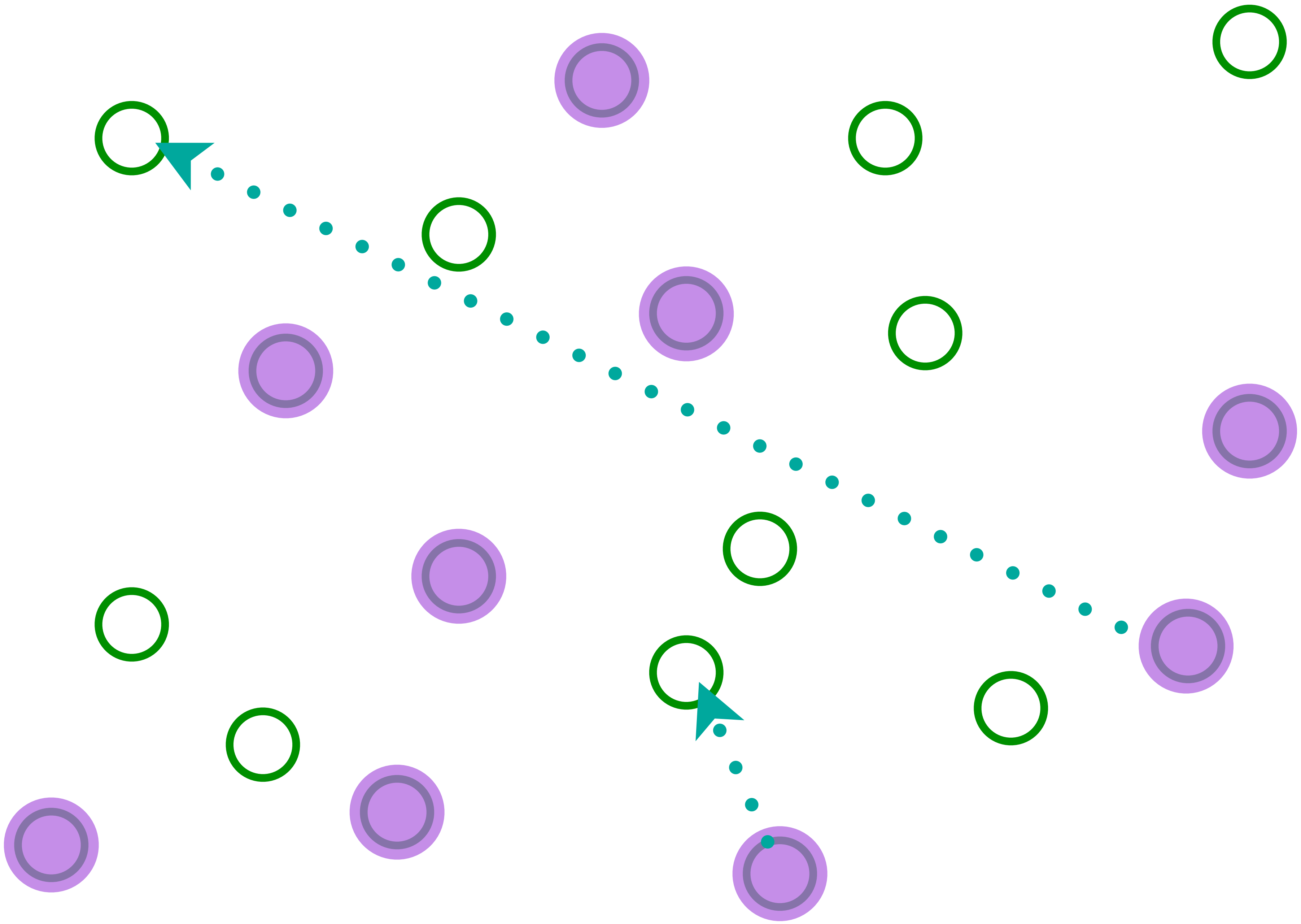

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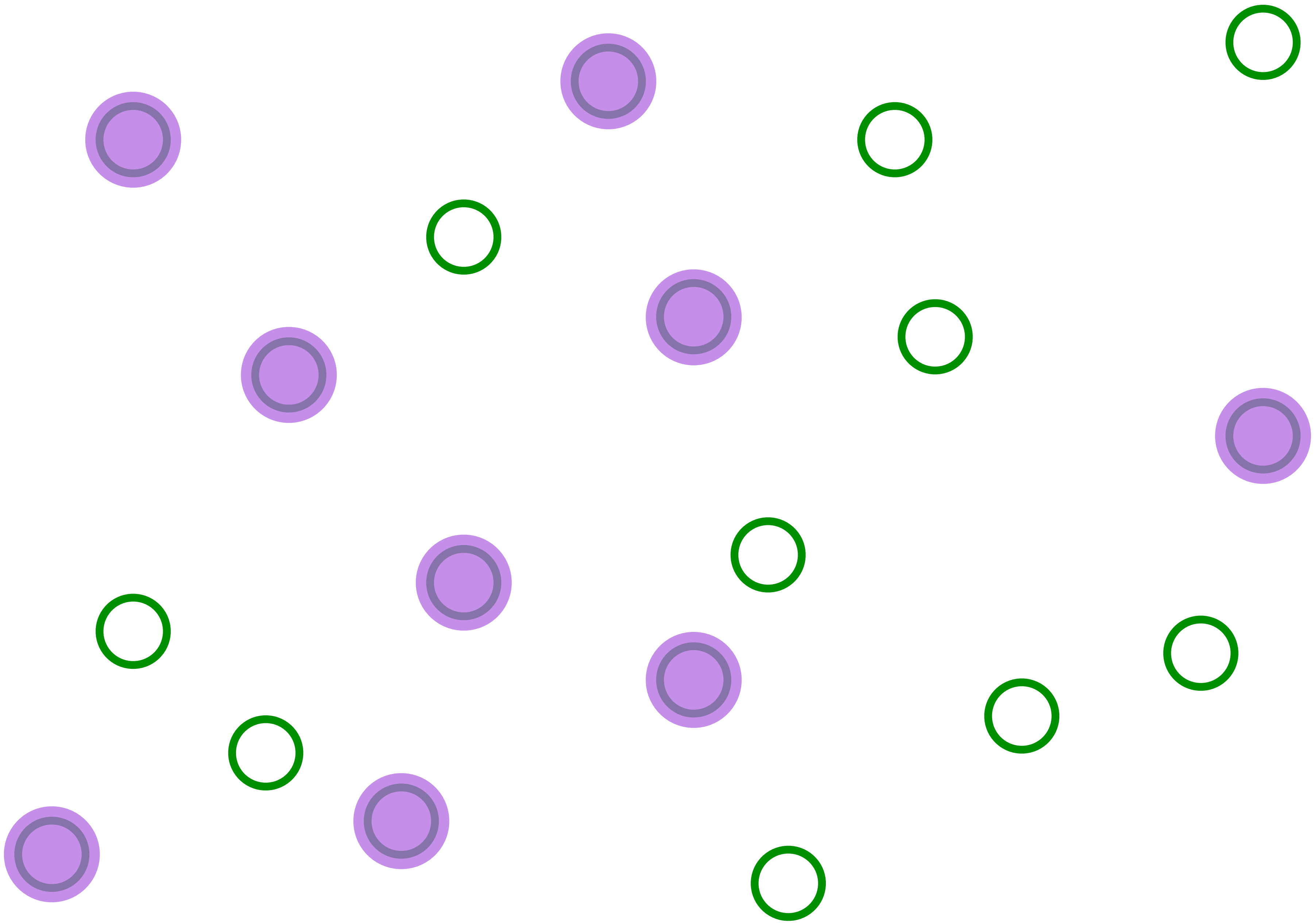

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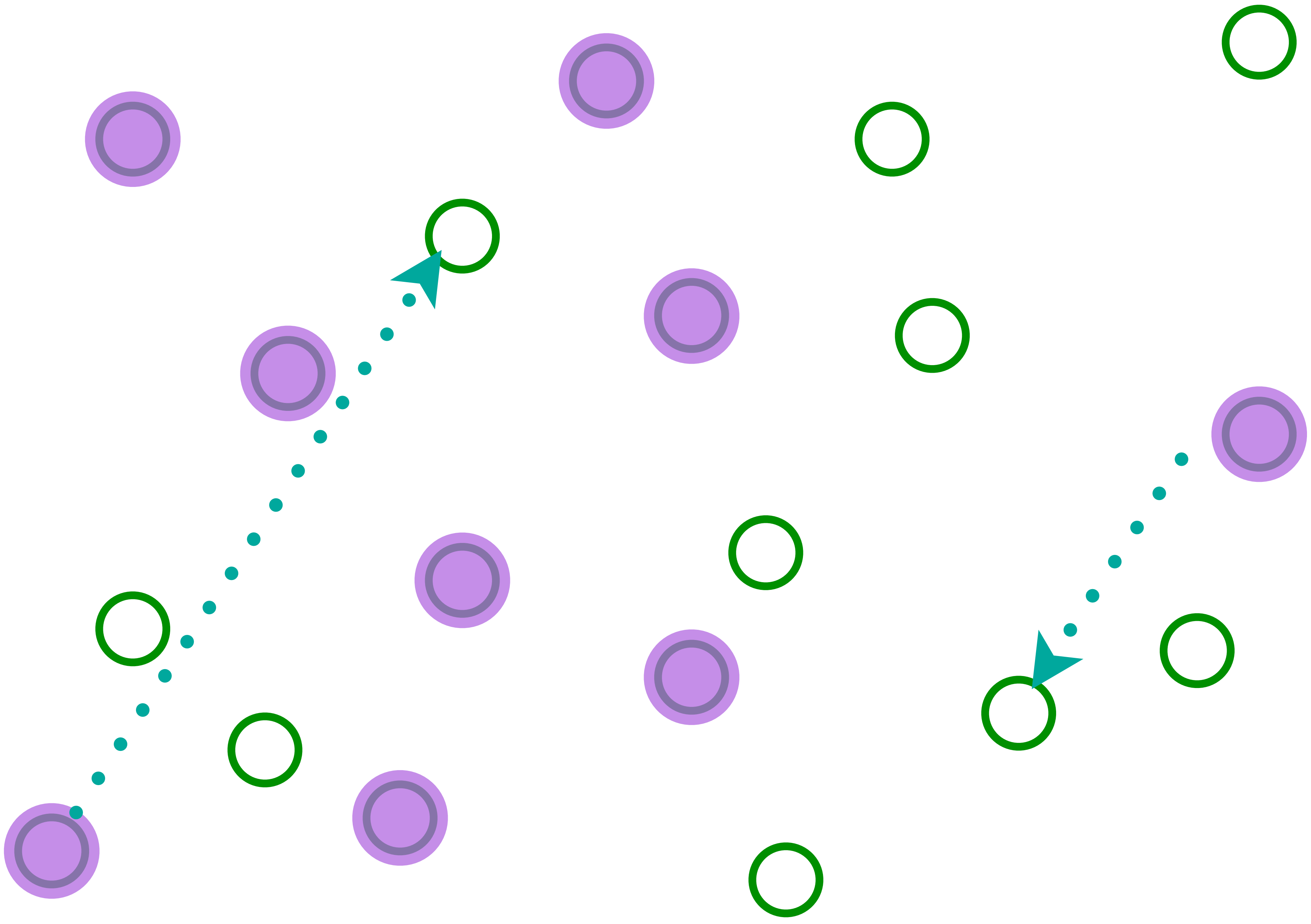

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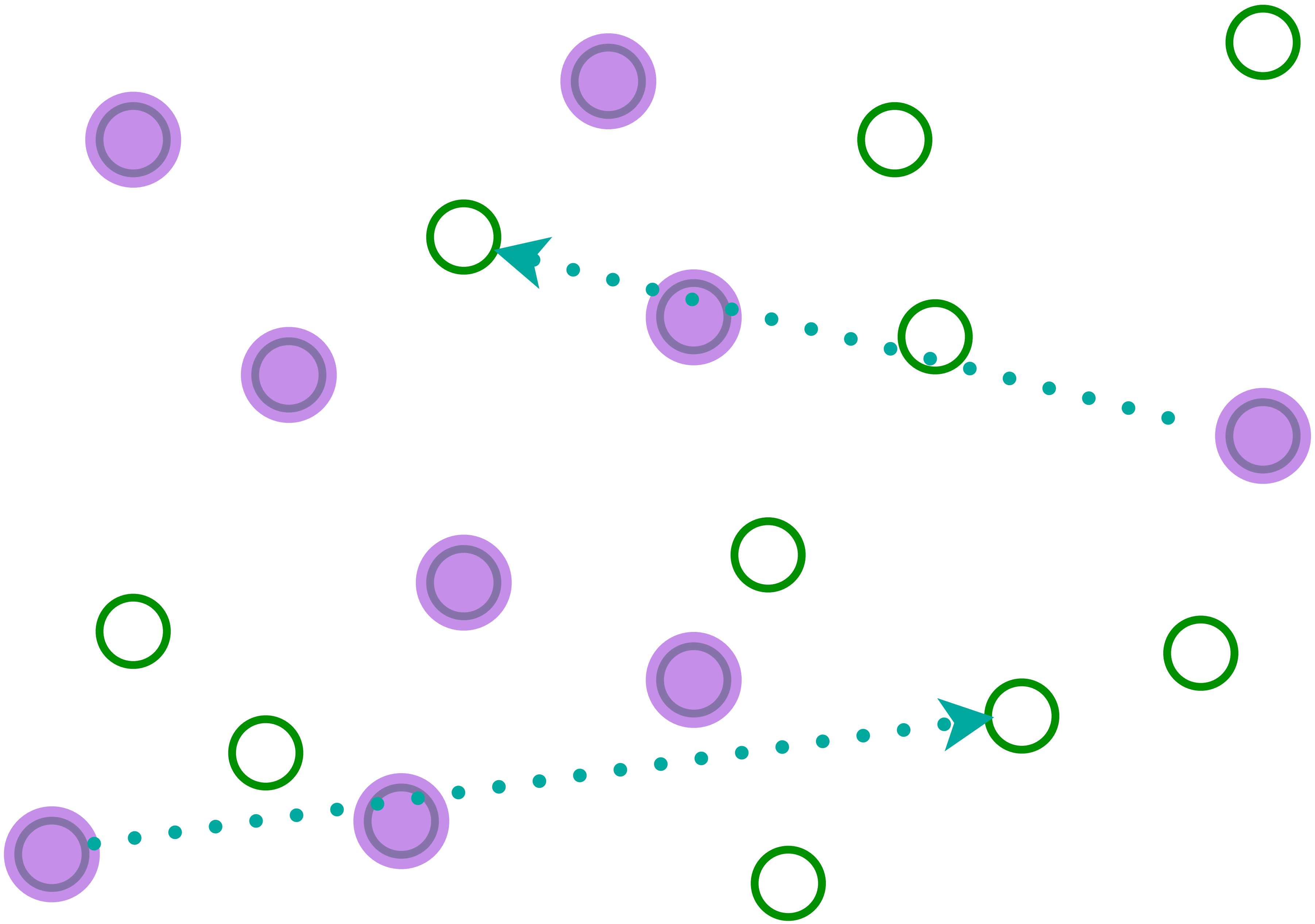

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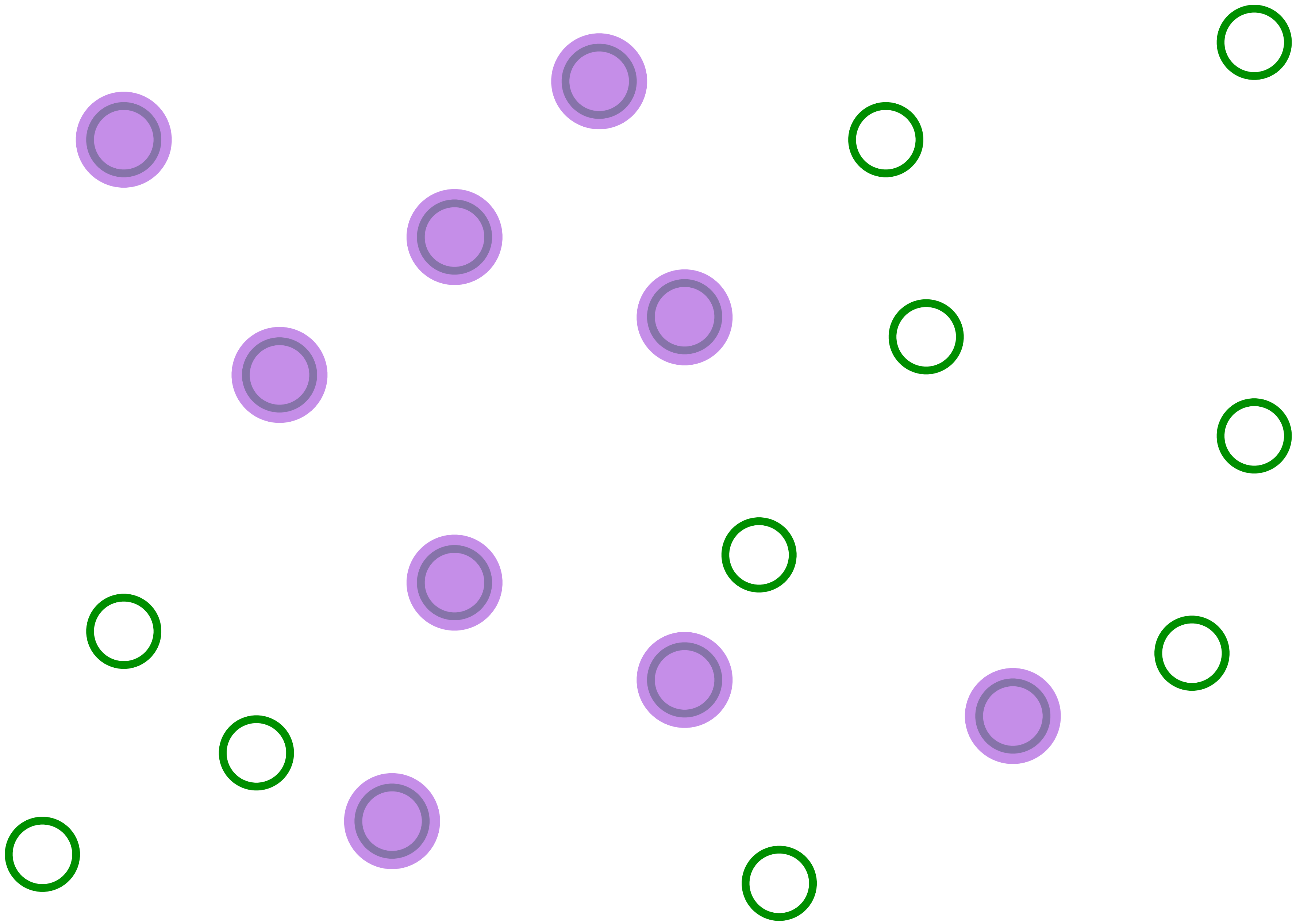

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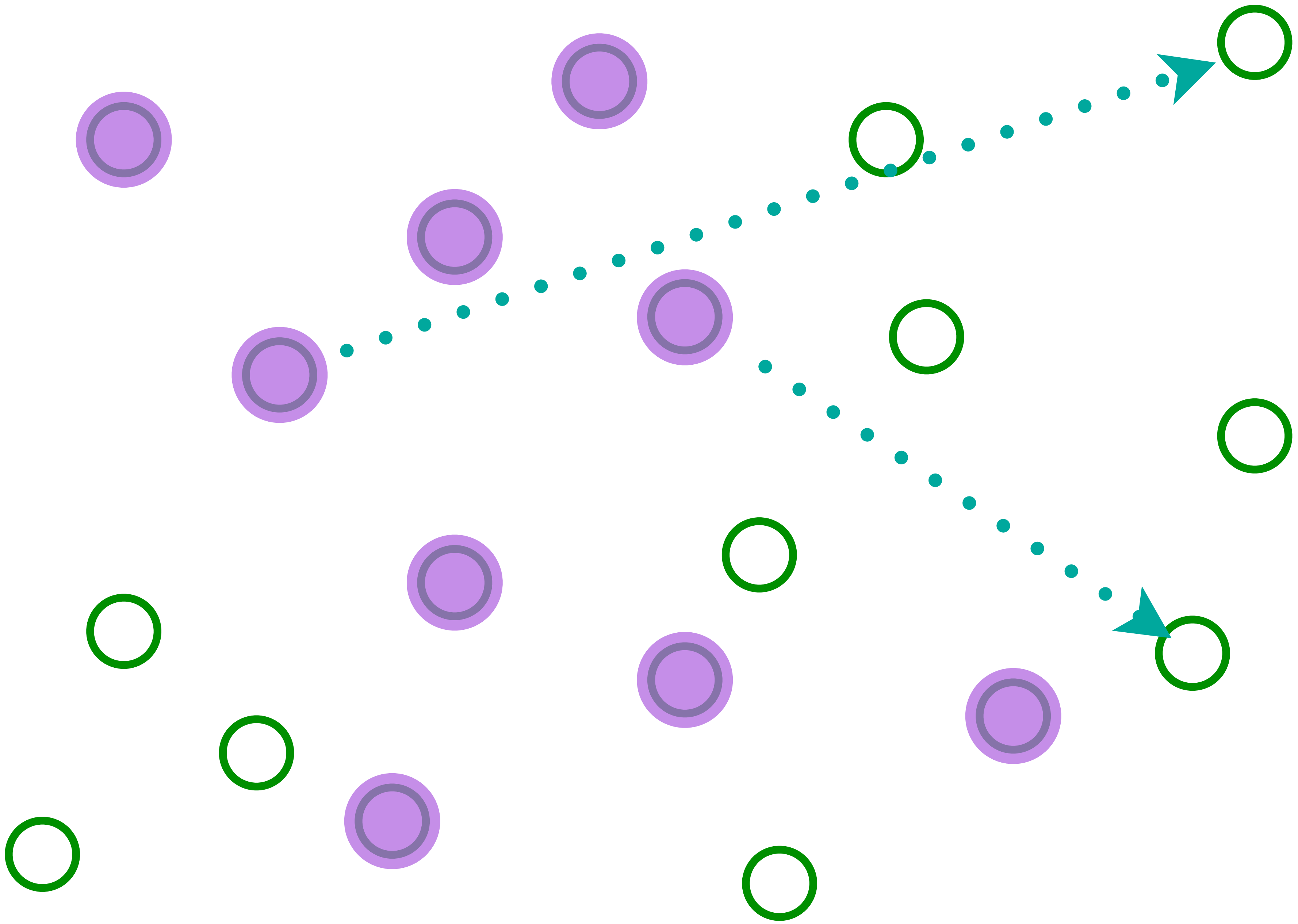

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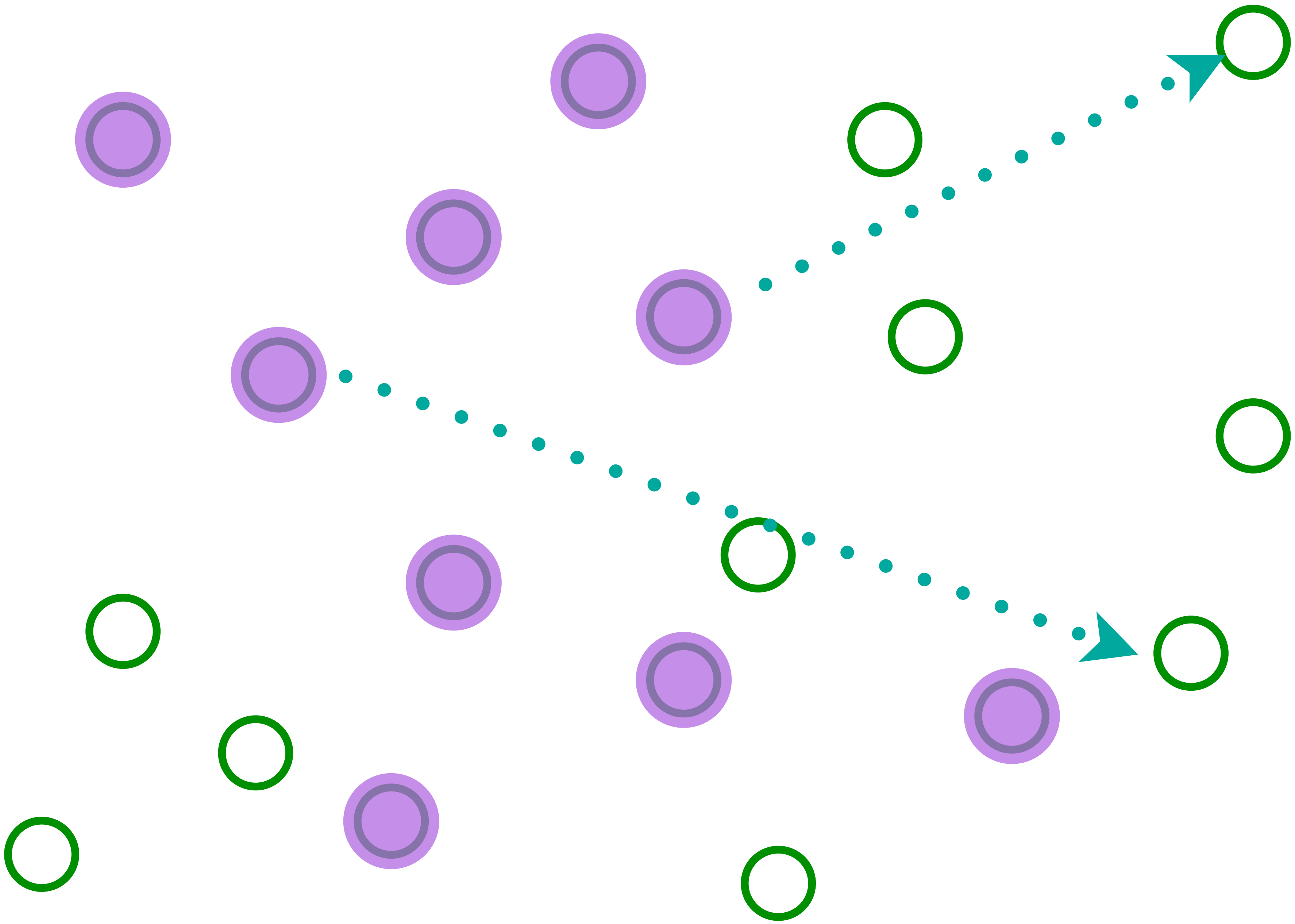

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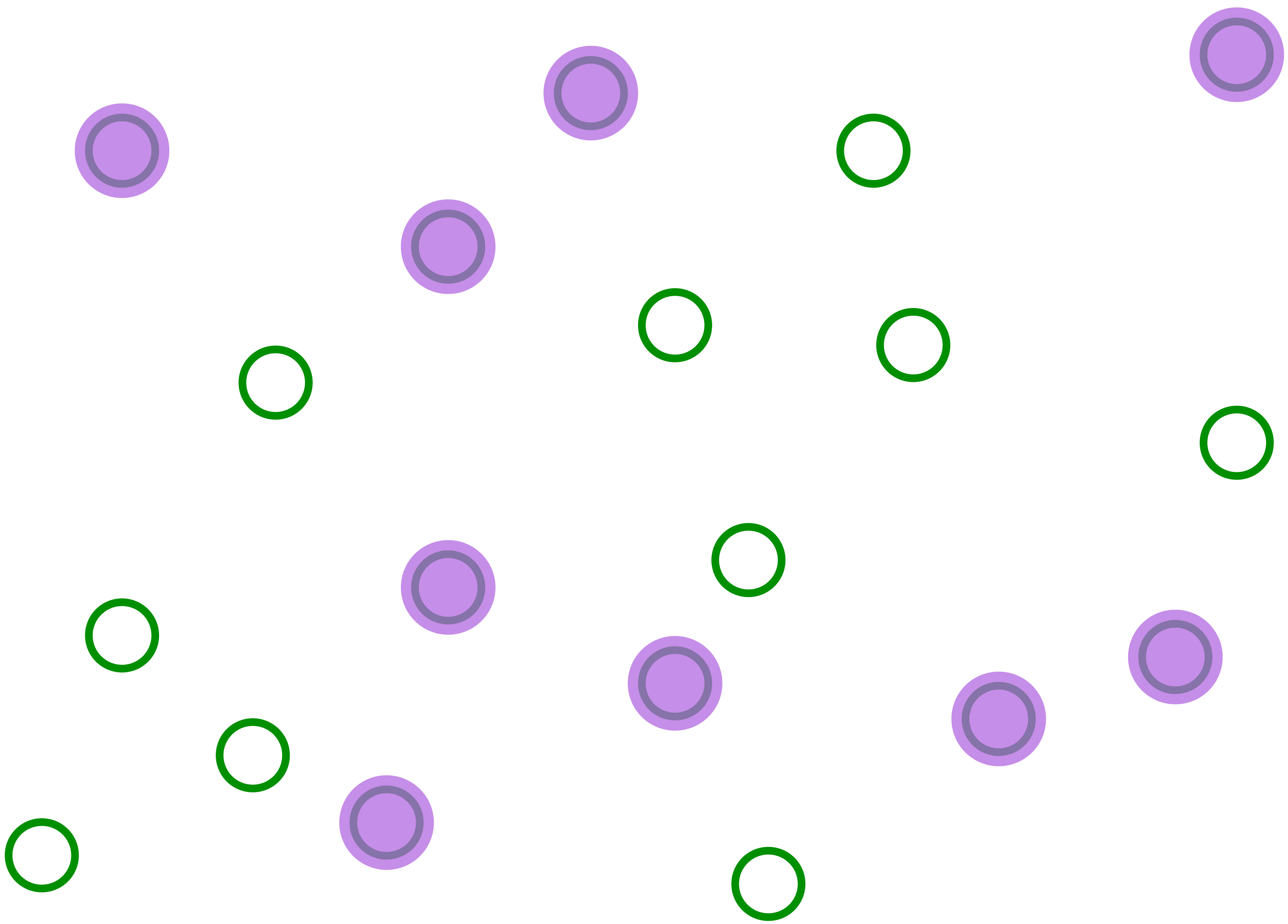

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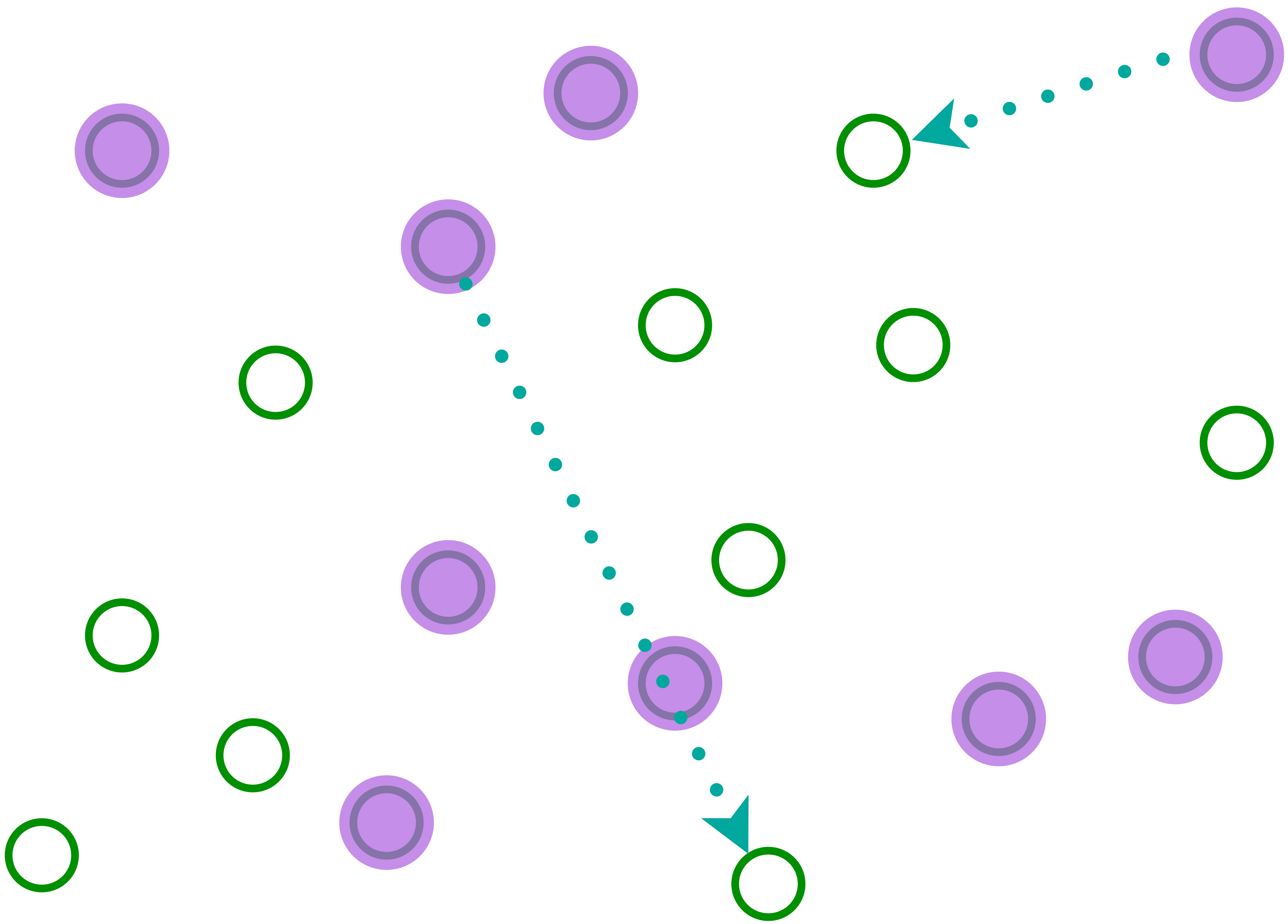

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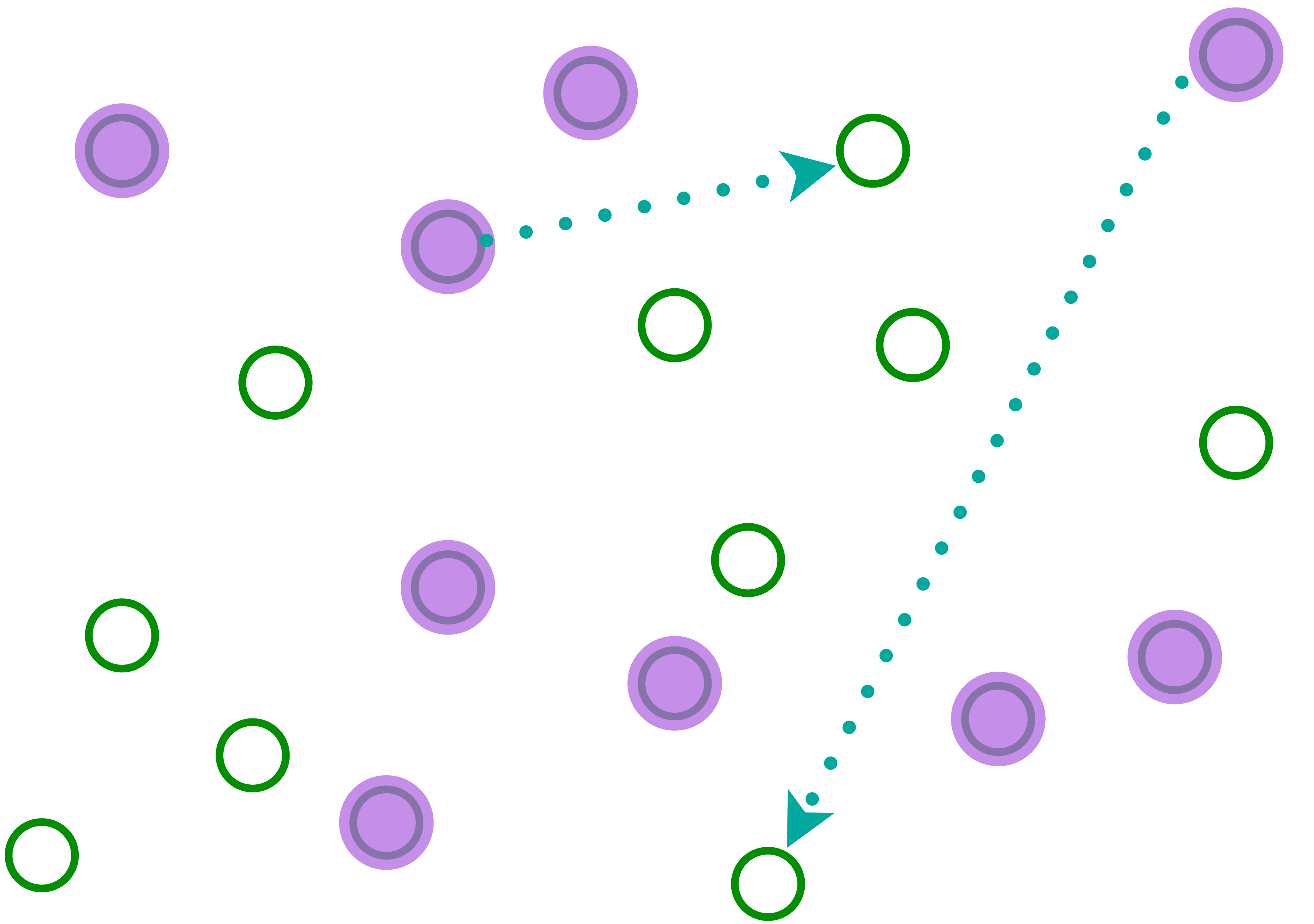

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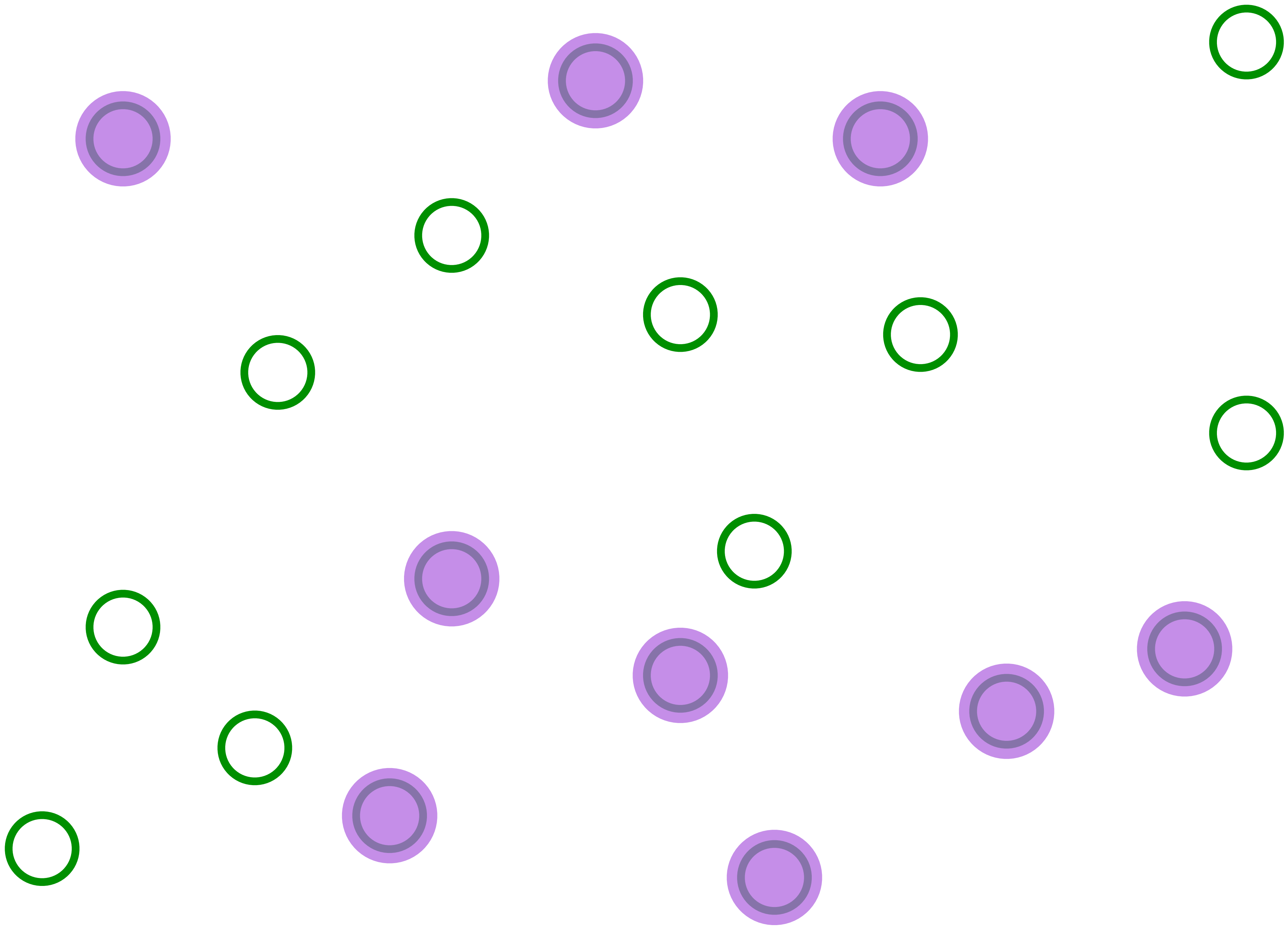

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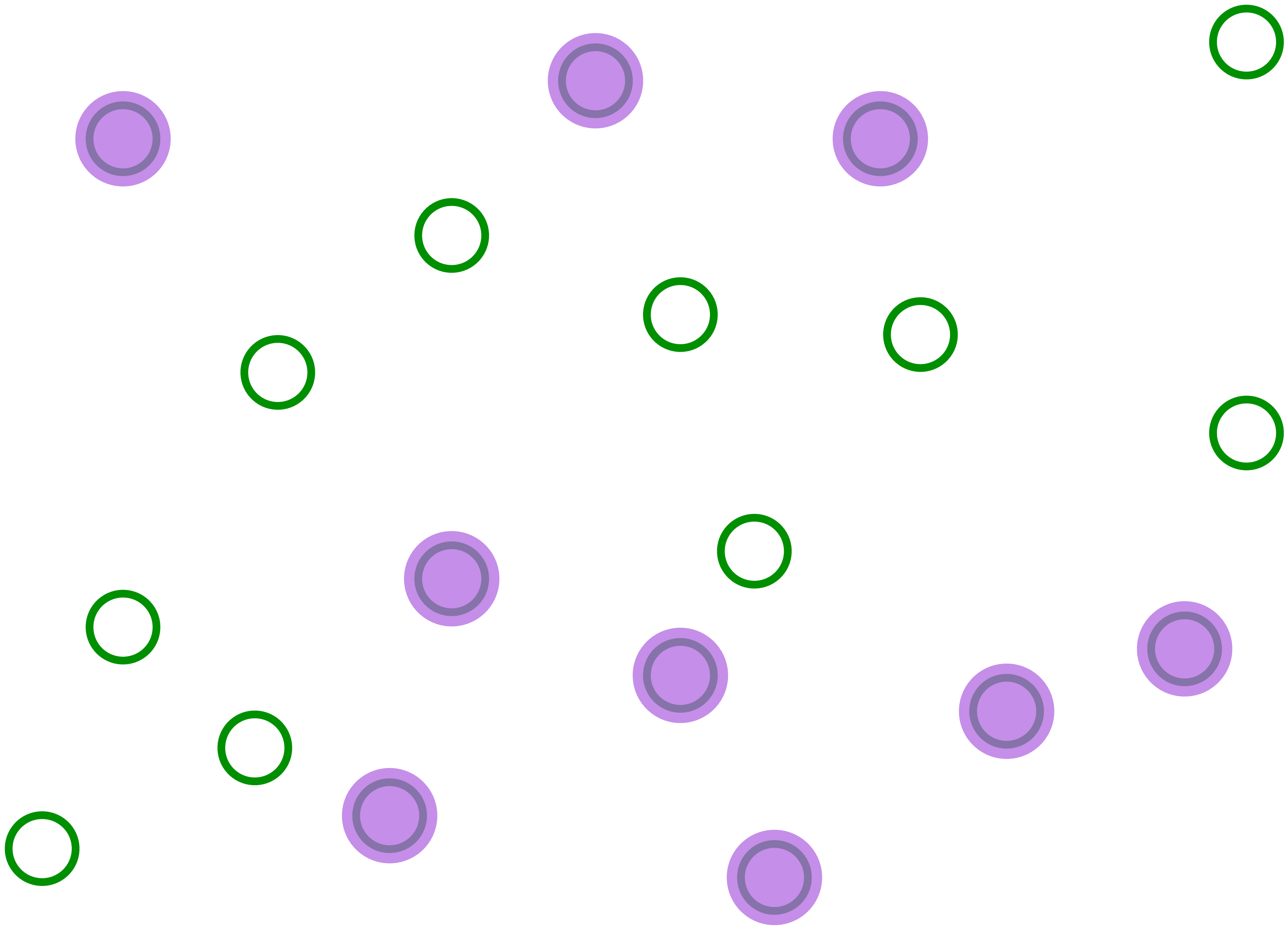

This describes both a strange metal and a black hole!

The SYK model

(See also: the "2-Body Random Ensemble" in nuclear physics; did not obtain the large *N* limit;
T.A. Brody, J. Flores, J.B. French, P.A. Mello, A. Pandey, and S.S.M. Wong, Rev. Mod. Phys. **53**, 385 (1981))

$$H = \frac{1}{(2N)^{3/2}} \sum_{i,j,k,\ell=1}^{N} U_{ij;k\ell}\, c_i^\dagger c_j^\dagger c_k c_\ell - \mu \sum_i c_i^\dagger c_i$$

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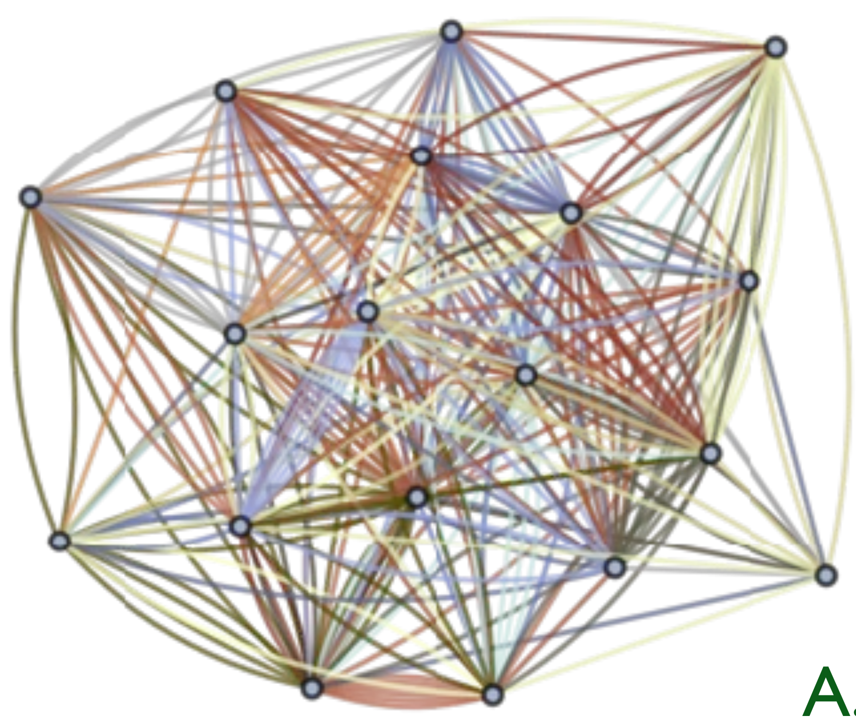
$U_{ij;k\ell}$ are independent random variables with $\overline{U_{ij;k\ell}} = 0$ and $\overline{|U_{ij;k\ell}|^2} = U^2$
$N \to \infty$ yields critical strange metal.



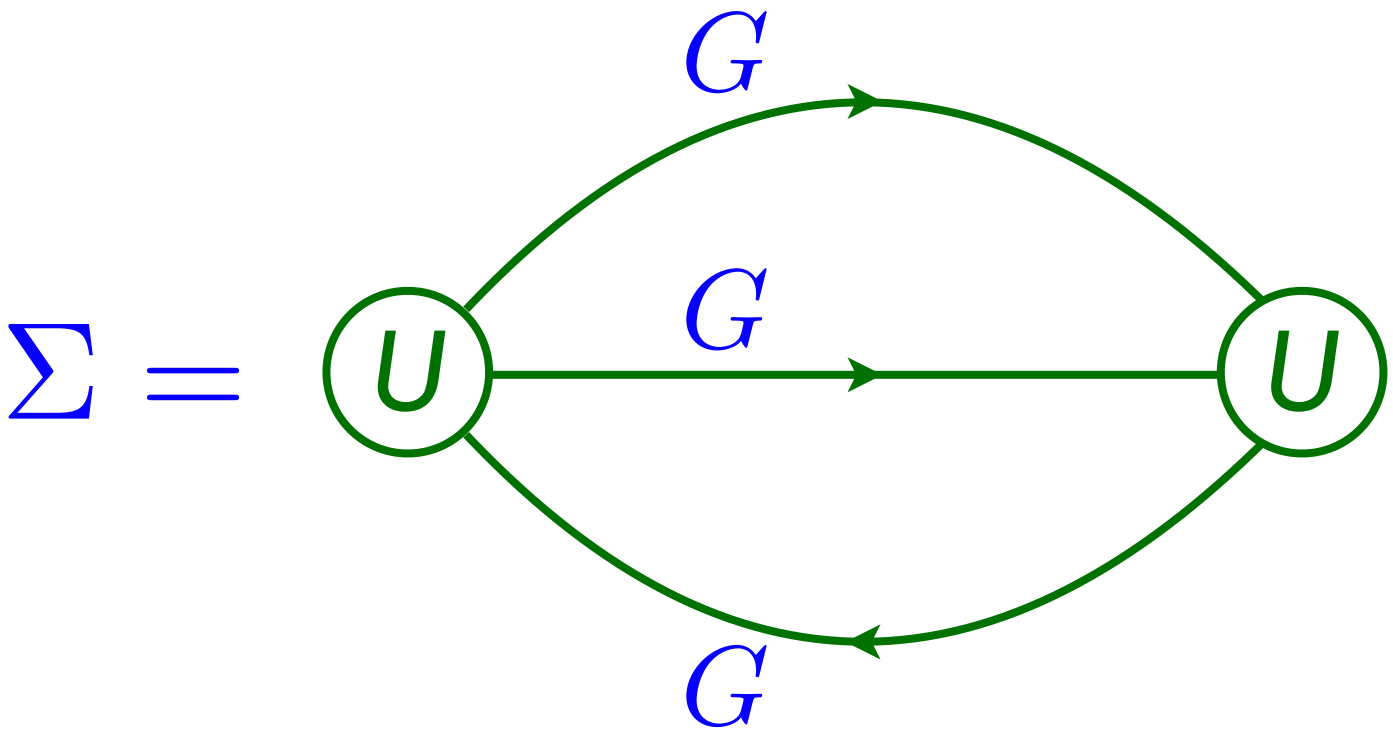
S. Sachdev and J. Ye, PRL **70**, 3339 (1993)

A. Kitaev, unpublished; S. Sachdev, PRX **5**, 041025 (2015)

The SYK model

Feynman graph expansion in $U_{ijk\ell}$, and graph-by-graph average, yields exact equations in the large N limit:

$$G(i\omega) = \frac{1}{i\omega + \mu - \Sigma(i\omega)} \quad , \quad \Sigma(\tau) = -U^2 G^2(\tau) G(-\tau)$$
$$G(\tau = 0^-) = \mathcal{Q}.$$



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Low frequency analysis shows that the solutions must be gapless and obey

$$\Sigma(z) = \mu - \frac{e^{i(\pi/4+\theta)}}{A}\sqrt{z} + \ldots \quad , \quad G(z) = \frac{A e^{-i(\pi/4+\theta)}}{\sqrt{z}}$$

where $A = (\pi/U^2 \cos(2\theta))^{1/4}$. The value of θ is universally related to $\mathcal{Q}$ by a Luttinger-Ward functional analysis similar to that used to establish the Luttinger theorem of Fermi liquid theory:

$$\boxed{\mathcal{Q} = \frac{1}{2} - \frac{\theta}{\pi} - \frac{\sin(2\theta)}{4}}$$

S. Sachdev and J. Ye, Phys. Rev. Lett. **70**, 3339 (1993)
A. Georges, O. Parcollet, and S. Sachdev, PRB **63**, 134406 (2001)

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At $T > 0$, we obtain a solution with a conformal structure

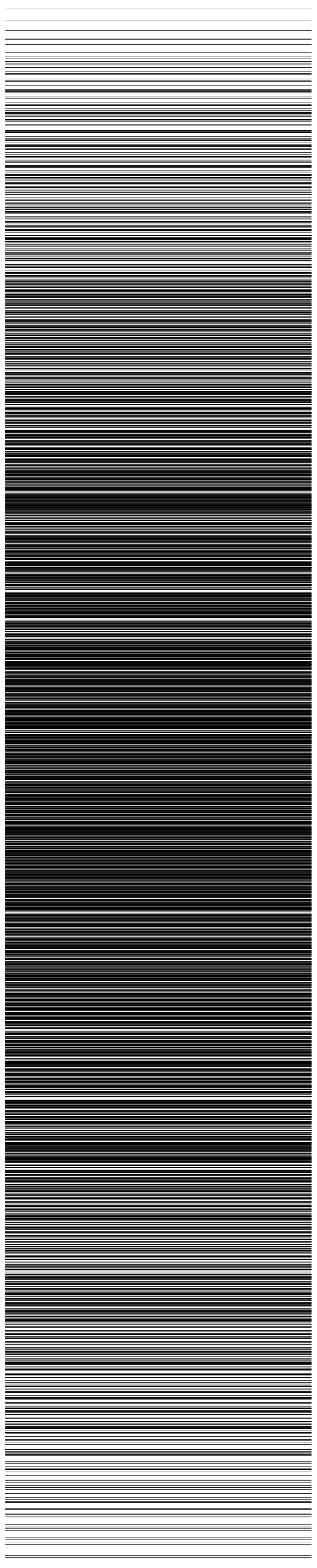
$$G(\tau) = -A \frac{e^{-2\pi\mathcal{E}T\tau}}{\sqrt{1 + e^{-4\pi\mathcal{E}}}} \left(\frac{T}{\sin(\pi T \tau)} \right)^{1/2} \quad , \quad 0 < \tau < 1/T \,,$$

where the 'particle-hole asymmetry' is determined by $\mathcal{E}$

$$\boxed{e^{2\pi\mathcal{E}} = \frac{\sin(\pi/4 + \theta)}{\sin(\pi/4 - \theta)}}\,.$$

A. Georges and O. Parcollet PRB **59**, 5341 (1999)
S. Sachdev, PRX **5**, 041025 (2015)

The SYK model



There are 2^N many body levels with energy E. Shown are all values of E for a single cluster of size $N = 12$. The $T \to 0$ state has an entropy $S_{GPS} = N s_0$, where $s_0 < \ln 2$ is determined by integrating

$$\boxed{\frac{ds_0}{d\mathcal{Q}} = 2\pi\mathcal{E}\,.}$$

At $\mathcal{Q} = 1/2$,

$$s_0 = \frac{G}{\pi} + \frac{\ln(2)}{4} = 0.464848\ldots$$

where G is Catalan's constant.

Many-body level spacing $\sim 2^{-N} = e^{-N\ln 2}$

Non-quasiparticle excitations with spacing $\sim e^{-Ns_0}$

GPS: A. Georges, O. Parcollet, and S. Sachdev, PRB **63**, 134406 (2001)

W. Fu and S. Sachdev, PRB **94**, 035135 (2016)

The SYK model

$$\Omega(T) - E_0 = N\left[-s_0 T - \frac{1}{2}(\gamma + 4\pi^2\mathcal{E}^2 K)T^2 + \mathcal{O}(T^3)\right] + 2T\ln\left(\frac{U}{T}\right)\ldots$$

is the grand potential, where $K = d\mathcal{Q}/d\mu \sim 1/U$ is the compressibility/N, $\gamma \sim 1/U$ will appear later in the co-efficient of the Schwarzian, and the N^0 term arises from fluctuations about the large N theory described by the Schwarzian.

The inversion from $\Omega(T)$ to the *many*-body density of states, $D(E)$, requires terms in $\Omega(T)$ which are exponentially small in N (not shown above) from the Schwarzian action, yielding terms which are not small in $D(E)$. We obtain

$$D(E) = \sum_{p=-\infty}^{\infty} e^{2\pi p \mathcal{E}}\, d\left(E - \frac{p^2}{2NK}\right)$$

where $N\mathcal{Q} + p$ is the integer fermion number, $d(E) = 0$ for $E < E_0$, and

$$\boxed{d(E) \sim \exp\left(N s_0\right)\sinh\left(\sqrt{2N\gamma(E - E_0)}\right) \quad , \quad E > E_0\ ,\ e^{-cN} \ll \gamma(E - E_0) \ll N}$$

There are exponentially more low energy states than for the quasiparticle case, and $D(E)$ self-averages down to energies exponentially small in N.

J. S. Cotler, G. Gur-Ari, M. Hanada, J. Polchinski, P. Saad, S. H. Shenker, D. Stanford, A. Streicher, and M. Tezuka, arXiv:1611.04650;
R. Davison, Wenbo Fu, A. Georges, Yingfei Gu, K. Jensen, S. Sachdev, arXiv.1612.00849 ;
A.M. Garcia-Garcia and J.J.M. Verbaarschot, arXiv:1701.06593; D. Bagrets, A. Altland, and A. Kamenev, arXiv:1702.08902;
D. Stanford and E. Witten, arXiv:1703.04612; A. Kitaev and S.J. Suh, arXiv:1711.08467; Yingfei Gu and S. Sachdev, unpublished.

A simple model of a metal with quasiparticles

The grand potential $\Omega(T)$ at low T is (from the Sommerfeld expansion)

$$\Omega(T) - E_0 = N\left(-\frac{\pi^2}{6}\rho_0 T^2 + \mathcal{O}(T^4)\right) + \ldots$$

where $\rho_0 \equiv \rho(0)$ is the *single* particle density of states at the Fermi level.
We can also define the *many* body density of states, $D(E)$, via

$$Z = e^{-\Omega(T)/T} = \int_{-\infty}^{\infty} D(E) e^{-E/T}$$

The inversion from $\Omega(T)$ to $D(E)$ has to performed with care (it need not commute with the $1/N$ expansion), and we obtain

$$\boxed{D(E) \sim \exp\left(\pi\sqrt{\frac{2N\rho_0(E-E_0)}{3}}\right) \quad , \quad E > E_0\ ,\ \frac{1}{N} \ll \rho_0(E-E_0) \ll N}$$

and $D(E) = 0$ for $E < E_0$. This is related to the asymptotic growth of the partitions of an integer, $p(n) \sim \exp(\pi\sqrt{2n/3})$. Near the lower bound, there are large sample-to-sample fluctuations due to variations in the lowest quasiparticle energies.

The SYK model

$$\Omega(T) - E_0 = N\left[-s_0 T - \frac{1}{2}(\gamma + 4\pi^2\mathcal{E}^2 K)T^2 + \mathcal{O}(T^3)\right] + 2T\ln\left(\frac{U}{T}\right)\ldots$$

is the grand potential, where $K = d\mathcal{Q}/d\mu \sim 1/U$ is the compressibility/N, $\gamma \sim 1/U$ will appear later in the co-efficient of the Schwarzian, and the N^0 term arises from fluctuations about the large N theory described by the Schwarzian.

The inversion from $\Omega(T)$ to the *many*-body density of states, $D(E)$, requires terms in $\Omega(T)$ which are exponentially small in N (not shown above) from the Schwarzian action, yielding terms which are not small in $D(E)$. We obtain

$$D(E) = \sum_{p=-\infty}^{\infty} e^{2\pi p\mathcal{E}}\, d\left(E - \frac{p^2}{2NK}\right)$$

where $N\mathcal{Q} + p$ is the integer fermion number, $d(E) = 0$ for $E < E_0$, and

$$d(E) \sim \boxed{\exp\left(N s_0\right)} \sinh\left(\sqrt{2N\gamma(E - E_0)}\right) \quad , \quad E > E_0\ ,\ e^{-cN} \ll \gamma(E - E_0) \ll N$$

There are exponentially more low energy states than for the quasiparticle case, and $D(E)$ self-averages down to energies exponentially small in N.

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D. Stanford and E. Witten, arXiv:1703.04612; A. Kitaev and S.J. Suh, arXiv:1711.08467; Yingfei Gu and S. Sachdev, unpublished.

The SYK model

No quasiparticles

- Rapid local thermal equilibration (of fermion correlators) in a 'Planckian' time

$$\tau_{\text{eq}} \sim \frac{\hbar}{k_B T} \quad , \quad \text{as } T \to 0.$$

A. Georges and O. Parcollet PRB **59**, 5341 (1999)
A. Eberlein, V. Kasper, S. Sachdev, and J. Steinberg, PRB **96**, 205123 (2017)

Established by solution of Schwinger-Keldysh equations for a quench.

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Established by solution of Schwinger-Keldysh equations for a quench.

- Presence of quasiparticles should slow down thermalization, so *all* quantum systems obey

S. Sachdev, *Quantum Phase Transitions*, Cambridge (1999)

$$\tau_{\rm eq} > C \frac{\hbar}{k_B T} \quad , \quad \text{as } T \to 0.$$

Absence of quasiparticles $\Leftrightarrow$ Fastest possible thermalization

Other quantum models without quasiparticles

- Rapid local thermal equilibration in a 'Planckian' time

$$\tau_{\text{eq}} \sim \frac{\hbar}{k_B T} \quad , \quad \text{as } T \to 0.$$

- Ising model in a transverse field in 2 dimensions at its quantum critical point, $g = g_c$. Described by the Wilson-Fisher fixed point of ϕ^4 quantum field theory in 2+1 dimensions

$$H = -\sum_{\langle ij \rangle} \sigma_i^z \, \sigma_j^z - g \sum_i \sigma_i^x$$

$\sigma_i^{x,z}$ are the Pauli operators on site i.

- Other strongly-coupled conformal field theories.

Absence of quasiparticles $\Leftrightarrow$ Fastest possible thermalization

1. Random matrix quasiparticle model

q=2, complex SYK

2. Matter without quasiparticles

q=4, complex SYK

3. The Schwarzian theory

4. Connections to black holes with AdS_2 horizons

5. Connections to strange metals

$$G(i\omega) = \frac{1}{\cancel{i\omega} + \cancel{\mu} - \Sigma(i\omega)} \quad , \quad \Sigma(\tau) = -U^2 G^2(\tau) G(-\tau)$$

$$\Sigma(z) = \cancel{\mu} - \frac{1}{A}\sqrt{z} + \dots \quad , \quad G(z) = \frac{A}{\sqrt{z}}$$

At frequencies $\ll U$, the $i\omega + \mu$ can be dropped, and without it equations are invariant under the reparametrization and gauge transformations. The singular part of the self-energy and the Green's function obey

$$\int_0^\beta d\tau_2 \, \Sigma_{\text{sing}}(\tau_1, \tau_2) G(\tau_2, \tau_3) = -\delta(\tau_1 - \tau_3)$$

$$\Sigma_{\text{sing}}(\tau_1, \tau_2) = -U^2 G^2(\tau_1, \tau_2) G(\tau_2, \tau_1)$$

A. Kitaev, 2015

S. Sachdev, PRX **5,** 041025 (2015)

$$\int_0^\beta d\tau_2\, \Sigma_(\tau_1, \tau_2) G(\tau_2, \tau_3) = -\delta(\tau_1 - \tau_3)$$

$$\Sigma(\tau_1, \tau_2) = -U^2 G^2(\tau_1, \tau_2) G(\tau_2, \tau_1)$$

These equations are invariant under

$$\tau = f(\sigma)$$

$$G(\tau_1, \tau_2) = [f'(\sigma_1) f'(\sigma_2)]^{-1/4} \frac{g(\sigma_1)}{g(\sigma_2)} \widetilde{G}(\sigma_1, \sigma_2)$$

$$\Sigma(\tau_1, \tau_2) = [f'(\sigma_1) f'(\sigma_2)]^{-3/4} \frac{g(\sigma_1)}{g(\sigma_2)} \widetilde{\Sigma}(\sigma_1, \sigma_2)$$

where $f(\sigma)$ and $g(\sigma)$ are arbitrary functions.
By using $f(\sigma) = \tan(\pi T \sigma)/(\pi T)$ we can
now obtain the $T > 0$ solution from the $T = 0$ solution.

A. Kitaev, 2015
S. Sachdev, PRX **5,** 041025 (2015)

Let us write the large N saddle point solutions of S as

$$\begin{aligned} G_s(\tau_1 - \tau_2) &\sim (\tau_1 - \tau_2)^{-1/2} \\ \Sigma_s(\tau_1 - \tau_2) &\sim (\tau_1 - \tau_2)^{-3/2}. \end{aligned}$$

The saddle point will be invariant under a reperamaterization $f(\tau)$ when choosing $G(\tau_1, \tau_2) = G_s(\tau_1 - \tau_2)$ leads to a transformed $\widetilde{G}(\sigma_1, \sigma_2) = G_s(\sigma_1 - \sigma_2)$ (and similarly for Σ). It turns out this is true only for the SL(2, R) transformations under which

$$f(\tau) = \frac{a\tau + b}{c\tau + d} \quad , \quad ad - bc = 1.$$

So the (approximate) reparametrization symmetry is spontaneously broken down to SL(2, R) by the saddle point.

A. Kitaev

The Schwarzian theory of the SYK model

After introducing replicas $a = 1 \dots n$, and integrating out the disorder, the partition function can be written as

$$Z = \int \mathcal{D}c_{ia}(\tau) \exp\left[-\sum_{ia} \int_0^\beta d\tau \, c_{ia}^\dagger \left(\frac{\partial}{\partial\tau} - \mu\right) c_{ia} \right.$$
$$\left. - \frac{U^2}{4N^3} \sum_{ab} \int_0^\beta d\tau d\tau' \left| \sum_i c_{ia}^\dagger(\tau) c_{ib}(\tau') \right|^4 \right] .$$

For simplicity, we neglect the replica indices, and introduce the identity

$$1 = \int \mathcal{D}\Sigma(\tau_1, \tau_2) \exp\left[-N \int_0^\beta d\tau_1 d\tau_2 \Sigma(\tau_1, \tau_2) \left(G(\tau_2, \tau_1) \right.\right.$$
$$\left.\left. + \frac{1}{N} \sum_i c_i(\tau_2) c_i^\dagger(\tau_1) \right) \right] .$$

The Schwarzian theory of the SYK model

Then the partition function can be written as a path integral with an action S analogous to a Luttinger-Ward functional

$$Z = \int \mathcal{D}G(\tau_1, \tau_2)\mathcal{D}\Sigma(\tau_1, \tau_2) \exp(-NS)$$

$$S = \ln\det\left[\delta(\tau_1 - \tau_2)(\partial_{\tau_1} + \mu) - \Sigma(\tau_1, \tau_2)\right] + \int d\tau_1 d\tau_2 \Sigma(\tau_1, \tau_2)\left[G(\tau_2, \tau_1) + (U^2/2)G^2(\tau_2, \tau_1)G^2(\tau_1, \tau_2)\right]$$

At frequencies $\ll U$, the time derivative in the determinant is less important, and without it the path integral is invariant under the reparametrization and gauge transformations

A. Georges and O. Parcollet
PRB **59**, 5341 (1999)
A. Kitaev, 2015
S. Sachdev, PRX **5,** 041025 (2015)

$$\tau = f(\sigma)$$

$$G(\tau_1, \tau_2) = \left[f'(\sigma_1)f'(\sigma_2)\right]^{-1/4} \frac{g(\sigma_1)}{g(\sigma_2)} G(\sigma_1, \sigma_2)$$

$$\Sigma(\tau_1, \tau_2) = \left[f'(\sigma_1)f'(\sigma_2)\right]^{-3/4} \frac{g(\sigma_1)}{g(\sigma_2)} \Sigma(\sigma_1, \sigma_2)$$

where $f(\sigma)$ and $g(\sigma)$ are arbitrary functions.

The Schwarzian theory of the SYK model

Reparametrization and phase zero modes

We can write the path integral for the SYK model as

$$\mathcal{Z} = \int \mathcal{D}G(\tau_1, \tau_1)\mathcal{D}\Sigma(\tau_1, \tau_2)e^{-NS[G,\Sigma]}$$

for a known action $S[G, \Sigma]$. We find the saddle point, G_s, Σ_s, and only focus on the "Nambu-Goldstone" modes associated with breaking reparameterization and U(1) gauge symmetries by writing

$$G(\tau_1, \tau_2) = [f'(\tau_1)f'(\tau_2)]^{1/4} G_s(f(\tau_1) - f(\tau_2))e^{i\phi(\tau_1) - i\phi(\tau_2)}$$

(and similarly for Σ). Then the path integral is approximated by

$$\mathcal{Z} = \int \mathcal{D}f(\tau)\mathcal{D}\phi(\tau)e^{-NS_{\text{eff}}[f,\phi]}.$$

J. Maldacena and D. Stanford, arXiv:1604.07818;
R. Davison, Wenbo Fu, A. Georges, Yingfei Gu, K. Jensen, S. Sachdev, arXiv.1612.00849;
A. Gaikwad, L.K. Joshi, G. Mandal, and S.R. Wadia, arXiv:1802.07746;
Yingfei Gu and S. Sachdev, unpublished

The Schwarzian theory of the SYK model

Symmetry arguments, and explicit computations, show that the effective action is

$$S_{\rm eff}[f,\phi] = \frac{K}{2}\int_0^{1/T} d\tau (\partial_\tau \phi + i(2\pi \mathcal{E} T)\partial_\tau f)^2 - \frac{\gamma}{4\pi^2}\int_0^{1/T} d\tau \, \{\tan(\pi T f(\tau)), \tau\},$$

where $f(\tau)$ is a monotonic map from $[0, 1/T]$ to $[0, 1/T]$, the couplings K, γ, and $\mathcal{E}$ can be related to thermodynamic derivatives and we have used the Schwarzian:

$$\{g, \tau\} \equiv \frac{g'''}{g'} - \frac{3}{2}\left(\frac{g''}{g'}\right)^2 .$$

Specifically, an argument constraining the effective at $T = 0$ is

$$S_{\rm eff}\left[f(\tau) = \frac{a\tau + b}{c\tau + d}, \phi(\tau) = 0\right] = 0,$$

and this is origin of the Schwarzian.

J. Maldacena and D. Stanford, arXiv:1604.07818;
R. Davison, Wenbo Fu, A. Georges, Yingfei Gu, K. Jensen, S. Sachdev, arXiv.1612.00849;
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SYK models and black holes

- Black holes have an entropy proportional to their surface area, and a temperature, $T_H = \hbar c^3/(8\pi G M k_B)$.
- Black holes relax to thermal equilibrium in a 'Planckian' time $\sim \hbar/(k_B T_H) = 8\pi G M/c^3$.
- Black holes in $d+1$ spatial dimensions are similar to a quantum system without quasiparticles in d spatial dimensions.

Black holes

SYK models and black holes

PHYSICAL REVIEW LETTERS **105**, 151602 (2010)

Holographic Metals and the Fractionalized Fermi Liquid

Subir Sachdev

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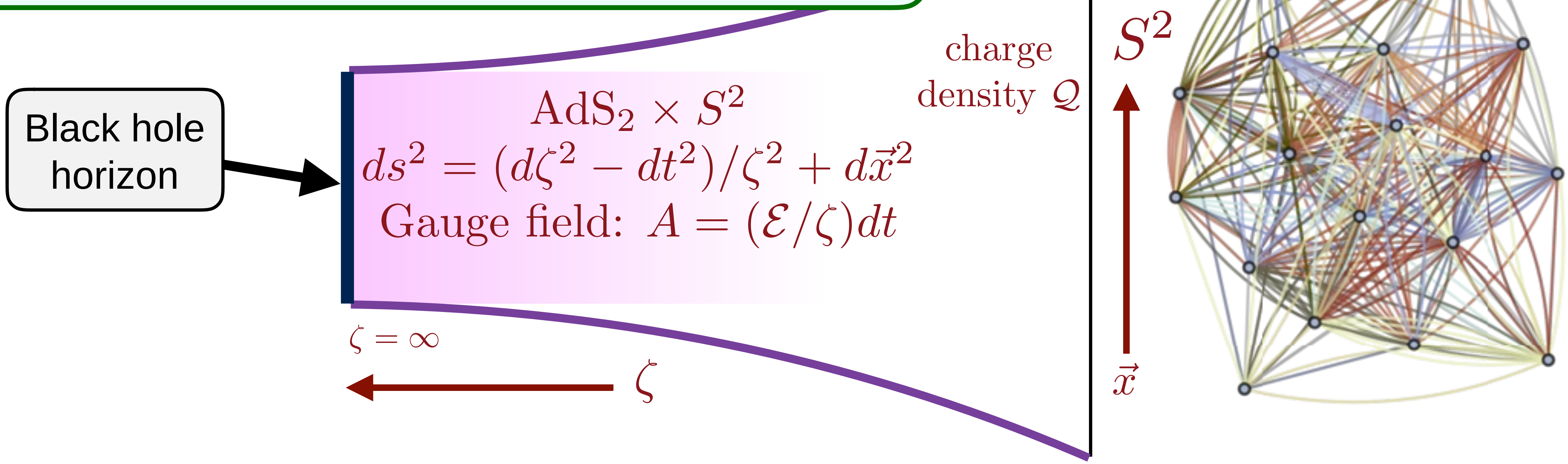
(Received 23 June 2010; published 4 October 2010)

We show that there is a close correspondence between the physical properties of holographic metals near charged black holes in anti–de Sitter (AdS) space, and the fractionalized Fermi liquid phase of the lattice Anderson model. The latter phase has a "small" Fermi surface of conduction electrons, along with a spin liquid of local moments. This correspondence implies that certain mean-field gapless spin liquids are states of matter at nonzero density realizing the near-horizon, $AdS_2 \times R^2$ physics of Reissner-Nordström black holes.

Bekenstein-Hawking entropy of AdS_2 horizon at $T = 0 \Leftrightarrow N s_0$ entropy of SYK model



SYK models and black holes

Connections of SYK to gravity and AdS_2 horizons

- Reparameterization and gauge invariance are the 'symmetries' of the Einstein-Maxwell theory of gravity and electromagnetism

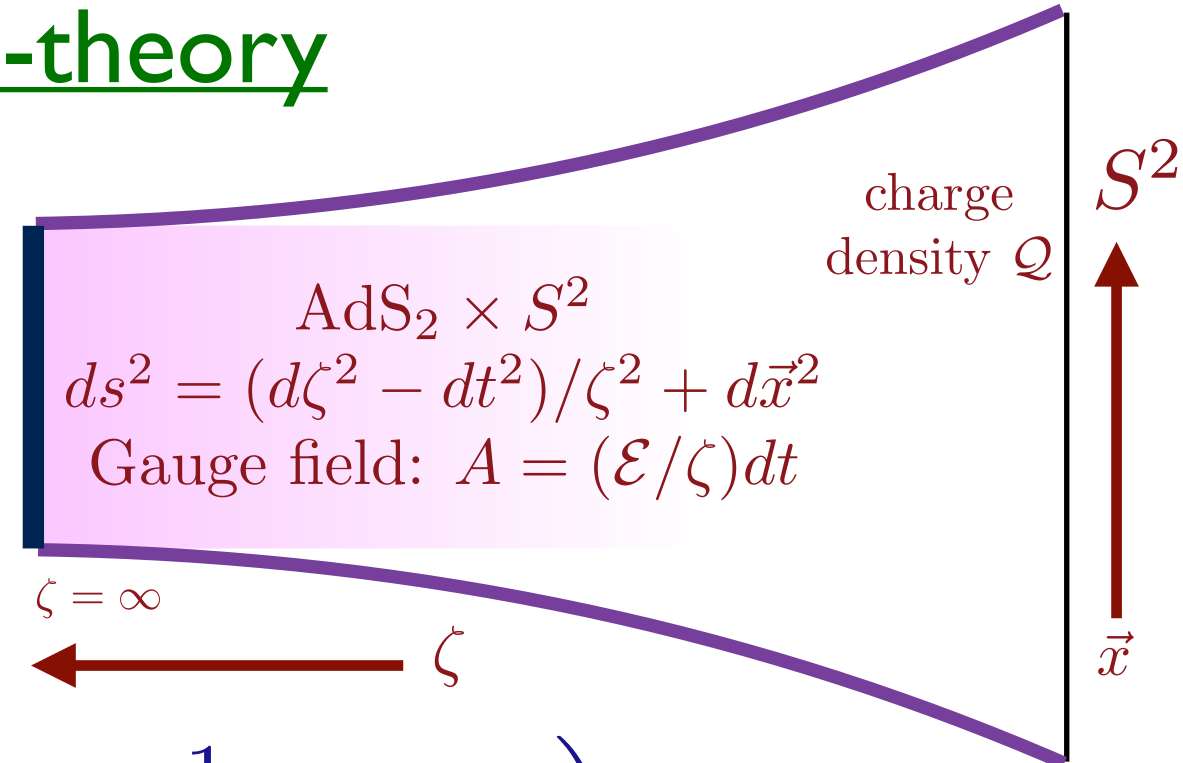
- SL(2,R) is the isometry group of AdS_2.

$ds^2 = (d\tau^2 + d\zeta^2)/\zeta^2$ is invariant under

$$\tau' + i\zeta' = \frac{a(\tau + i\zeta) + b}{c(\tau + i\zeta) + d}$$

with $ad - bc = 1$.

Einstein-Maxwell-theory



$$S_{4D} = \int d^4x \sqrt{-\hat{g}} \left(\hat{\mathcal{R}} + 6/L^2 - \frac{1}{4} \hat{F}_{\mu\nu} \hat{F}^{\mu\nu} \right),$$

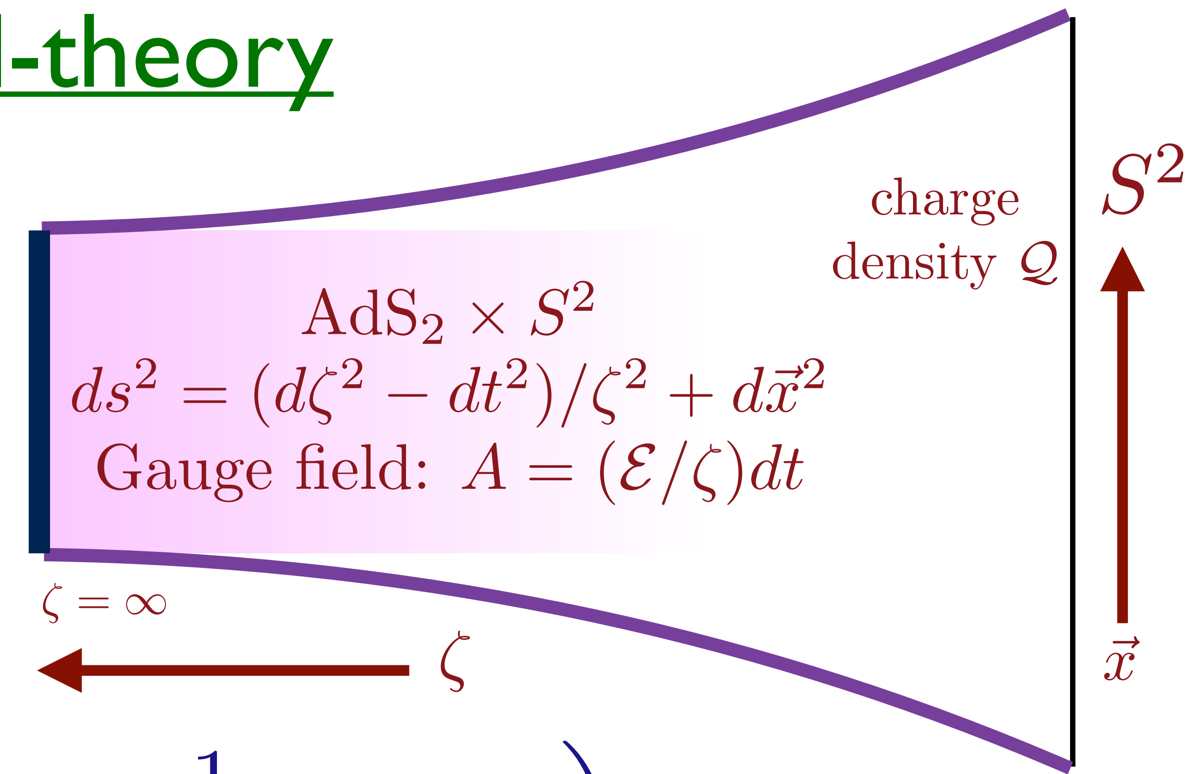
- Has Reissner-Nördstrom-AdS charged black hole solution, with charge density $\mathcal{Q}$, a near-horizon $\mathrm{AdS}_2 \times S^2$ geometry, and surface electric field $\mathcal{E}$.
- The Bekenstein-Hawking black hole entropy S_{4D} obeys the same relation as the SYK model

$$\boxed{\frac{\partial S_{4D}}{\partial \mathcal{Q}} = 2\pi \mathcal{E}\,,}$$

where $\mathcal{E}$ is identified from the spectral asymmetry of probe particle Green's functions in both cases.

S. Sachdev, PRX **5,** 041025 (2015)

Einstein-Maxwell-theory



P. Nayak, A. Shukla, R.M. Soni, S.P. Trivedi, and V. Vishal, arXiv:1802.09547;
A. Gaikwad, L.K. Joshi, G. Mandal, and S.R. Wadia, arXiv:1802.07746

$$S_{4D} = \int d^4x \sqrt{-\hat{g}} \left(\hat{\mathcal{R}} + 6/L^2 - \frac{1}{4} \hat{F}_{\mu\nu} \hat{F}^{\mu\nu} \right),$$

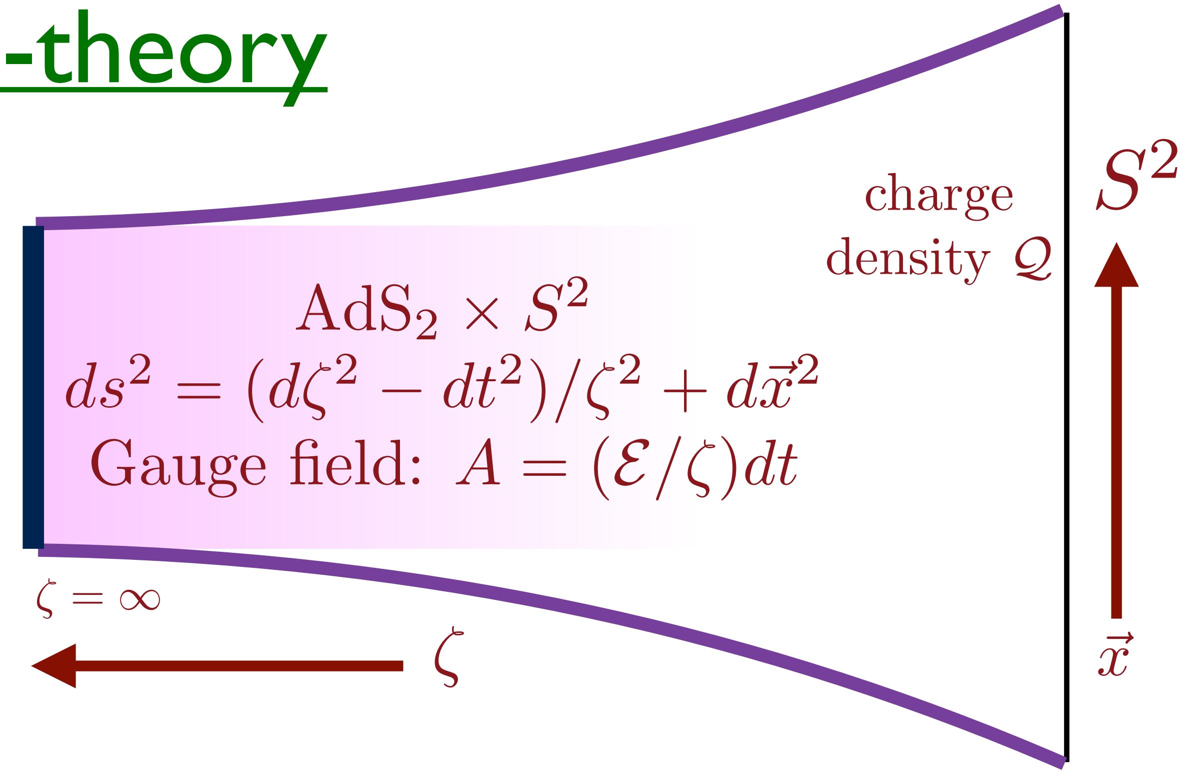
In the small black hole size limit, $T \ll 1/R$, where R is the radius of the black hole, the theory dimensionally reduces to an Einstein-Maxwell-dilaton theory in two dimensions (the Jackiw-Teitelbaum model), along with Maxwell term

$$S_{2D} = N s_0 + \int d^2x \sqrt{-g} \left(\Phi(\mathcal{R} - \Lambda) - \frac{Z(\Phi)}{4} F_{ab} F^{ab} \right).$$

The dilaton Φ represents the radial oscillations of the small black hole.

Einstein-Maxwell-theory

P. Nayak, A. Shukla, R.M. Soni, S.P. Trivedi, and V. Vishal, arXiv:1802.09547;
A. Gaikwad, L.K. Joshi, G. Mandal, and S.R. Wadia, arXiv:1802.07746



$$S_{2D} = N s_0 + \int d^2x \sqrt{-g} \left(\Phi(\mathcal{R} - \Lambda) - \frac{Z(\Phi)}{4} F_{ab} F^{ab} \right) .$$

There are no bulk quantum fluctuations of the metric in two-dimensional gravity, and there a further dimensional reduction to a $0+1$ dimensional theory representing fluctuations of the AdS_2 boundary: this $0+1$ dimensional turns out to be *precisely the Schwarzian theory obtained for the SYK model.*

J. Maldacena, D. Stanford, and Zhenbin Yang, arXiv:1606.01857;
K. Jensen, arXiv:1605.06098;
J. Engelsoy, T.G. Mertens, and H. Verlinde, arXiv:1606.03438

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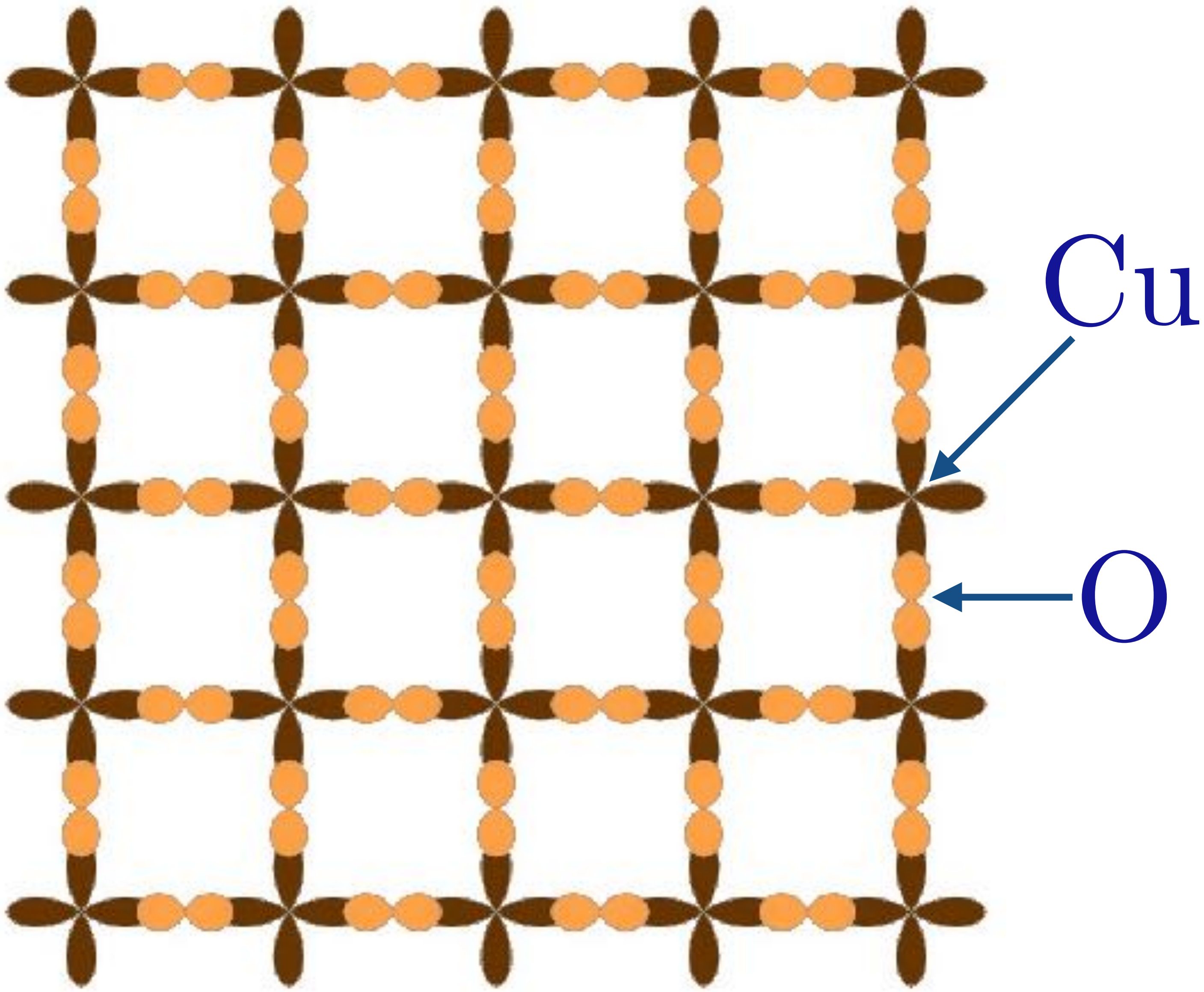
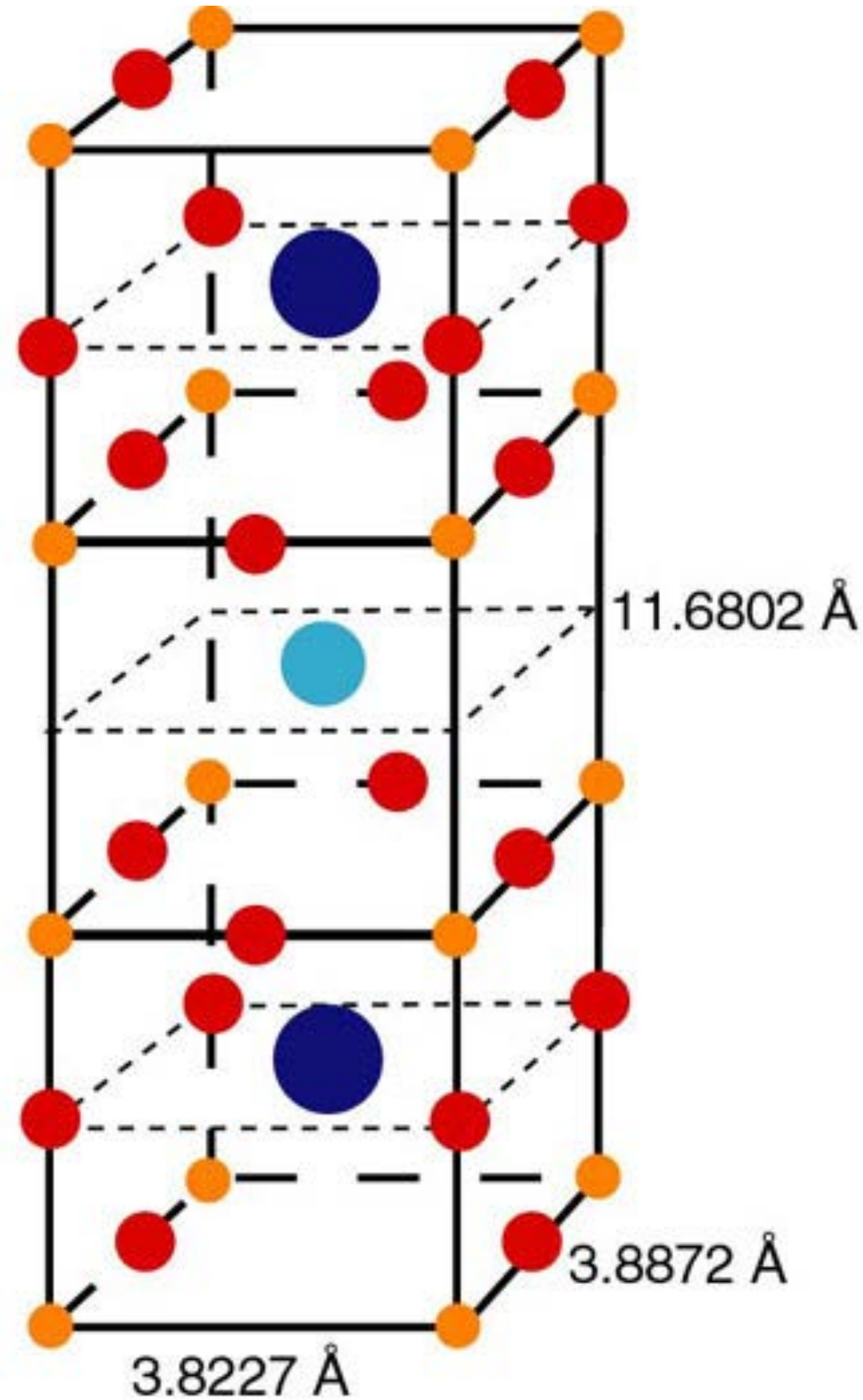
High temperature superconductors

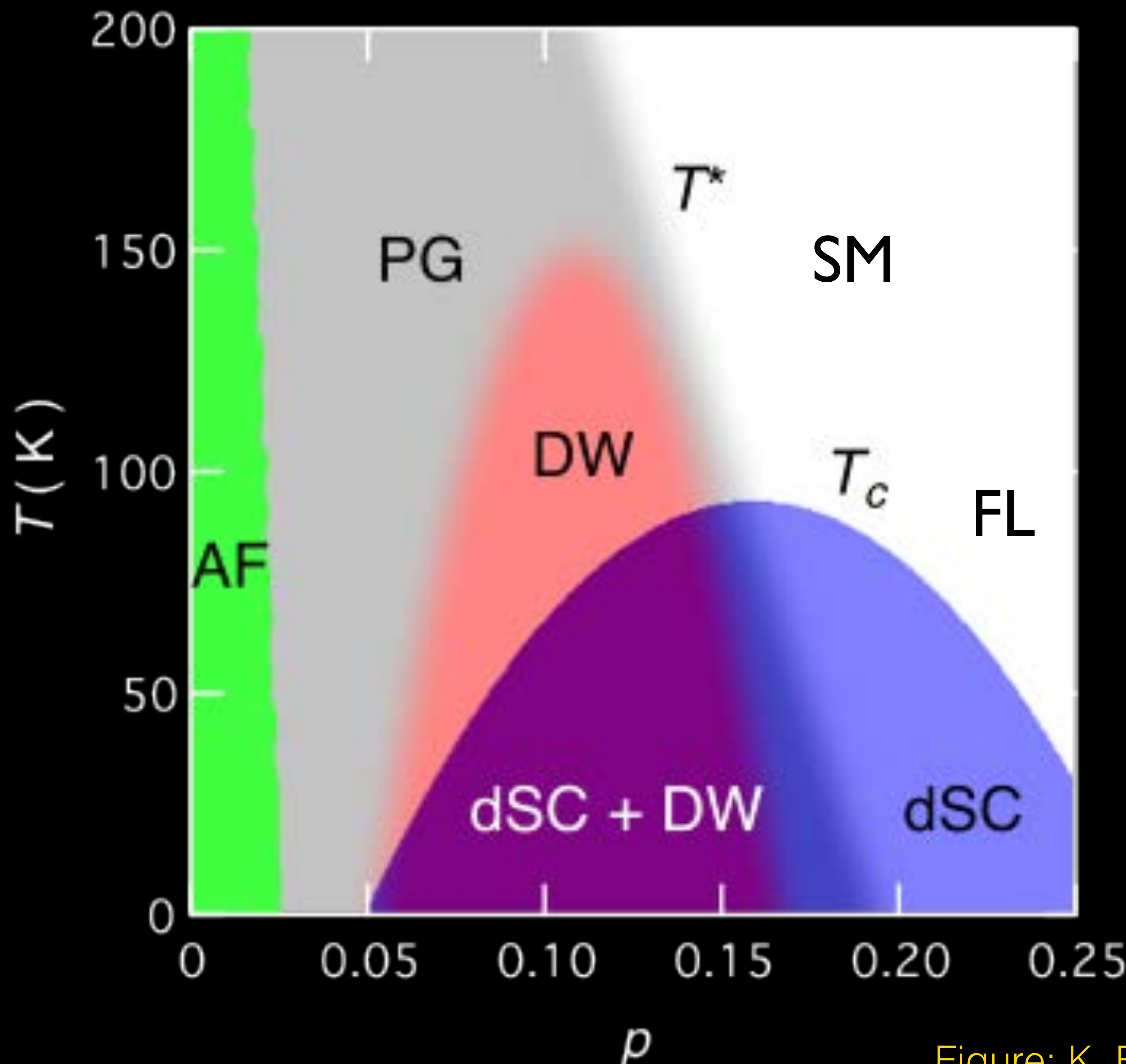


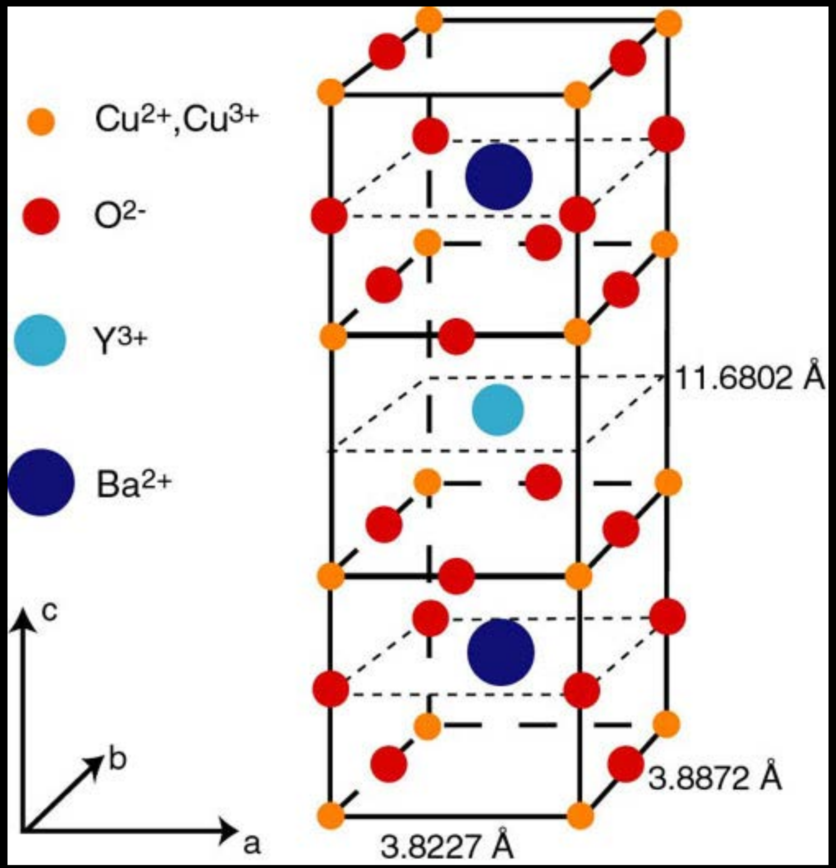
CuO_2 plane

Described by a Hubbard model on the Cu sites



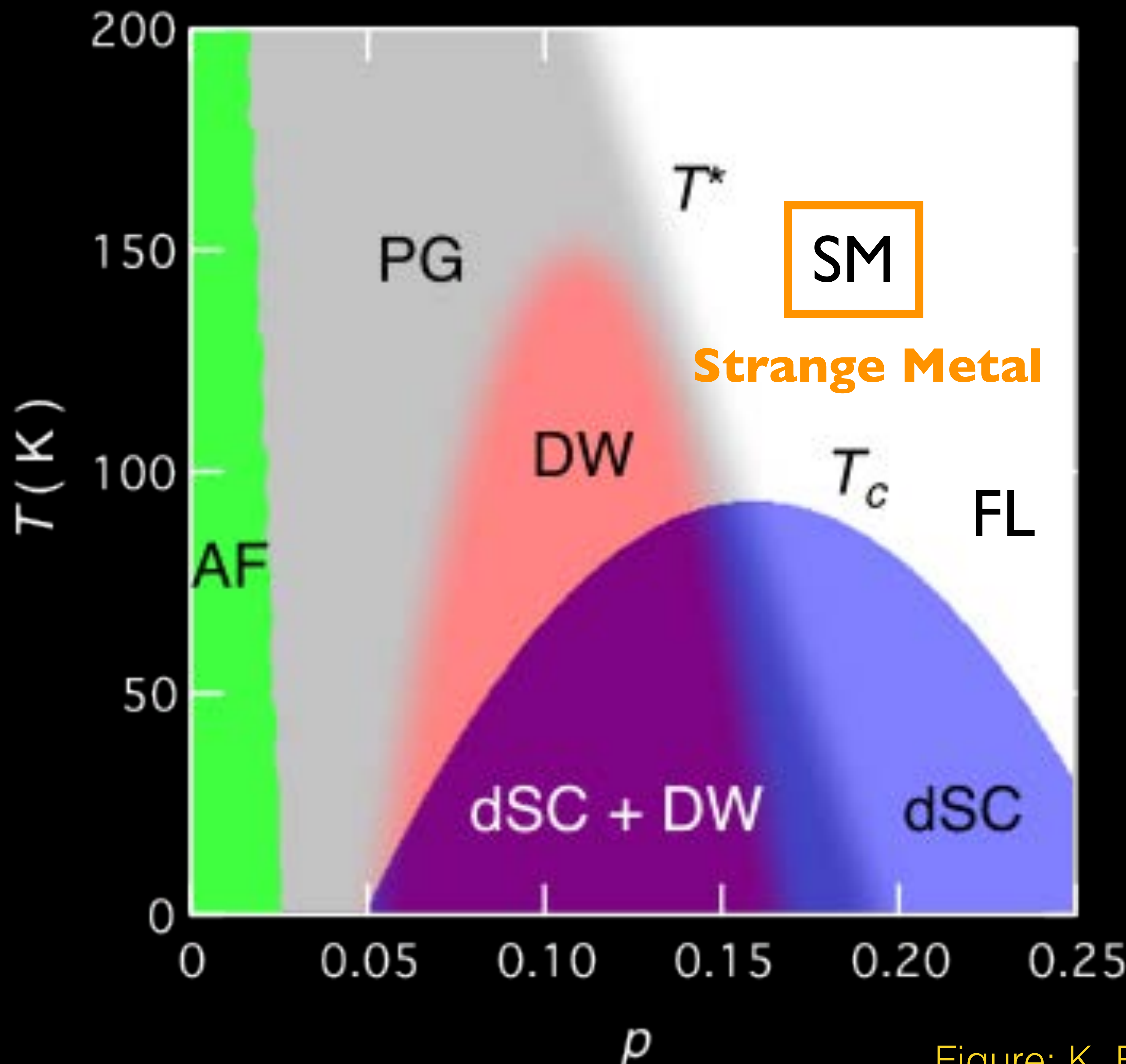
$YBa_2Cu_3O_{6+x}$

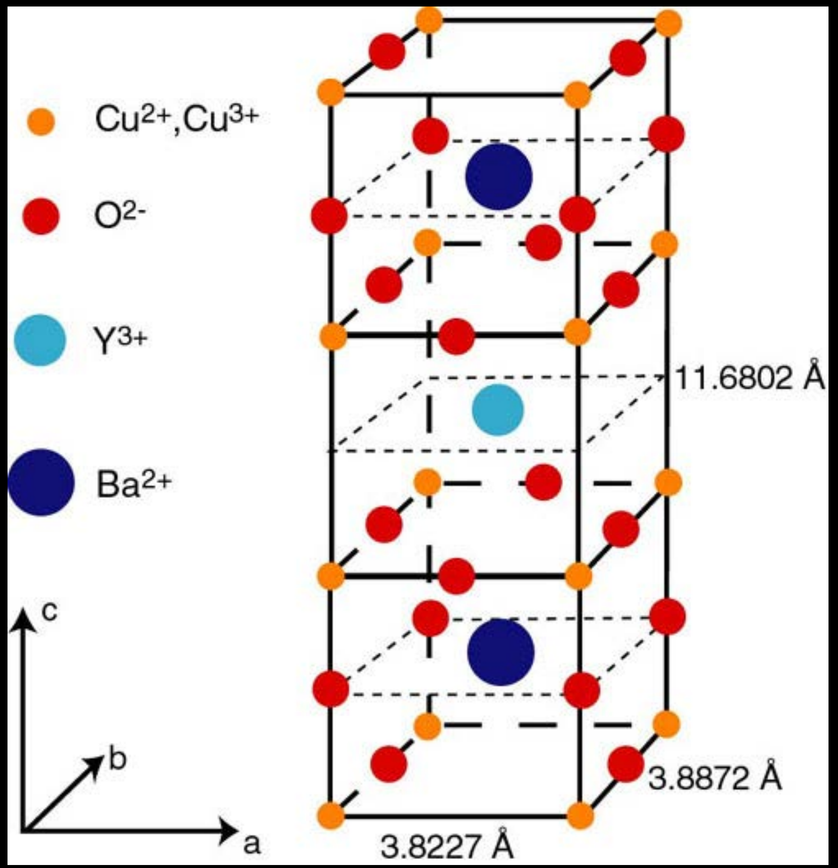




$YBa_2Cu_3O_{6+x}$

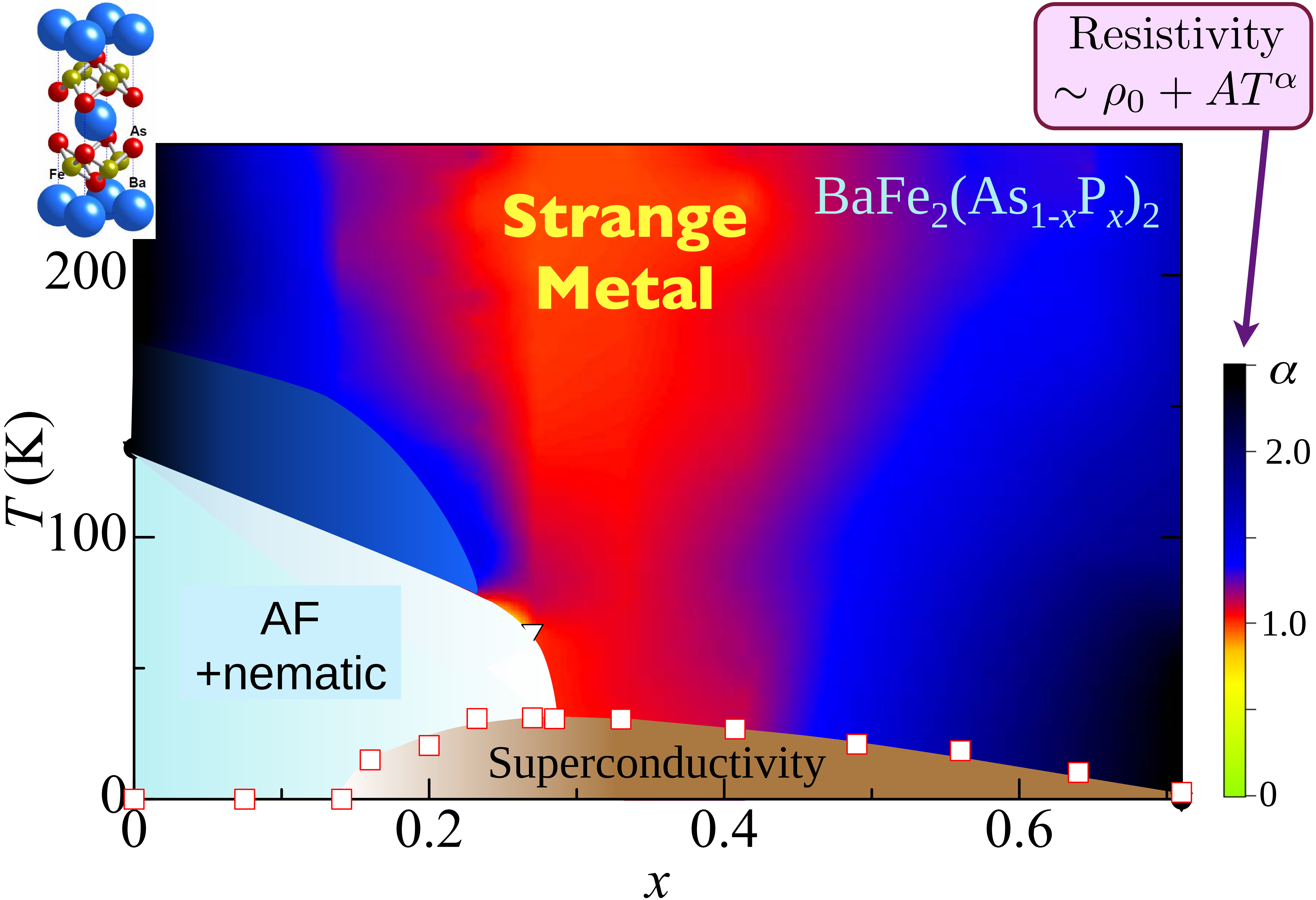
Figure: K. Fujita and J. C. Seamus Davis





$YBa_2Cu_3O_{6+x}$

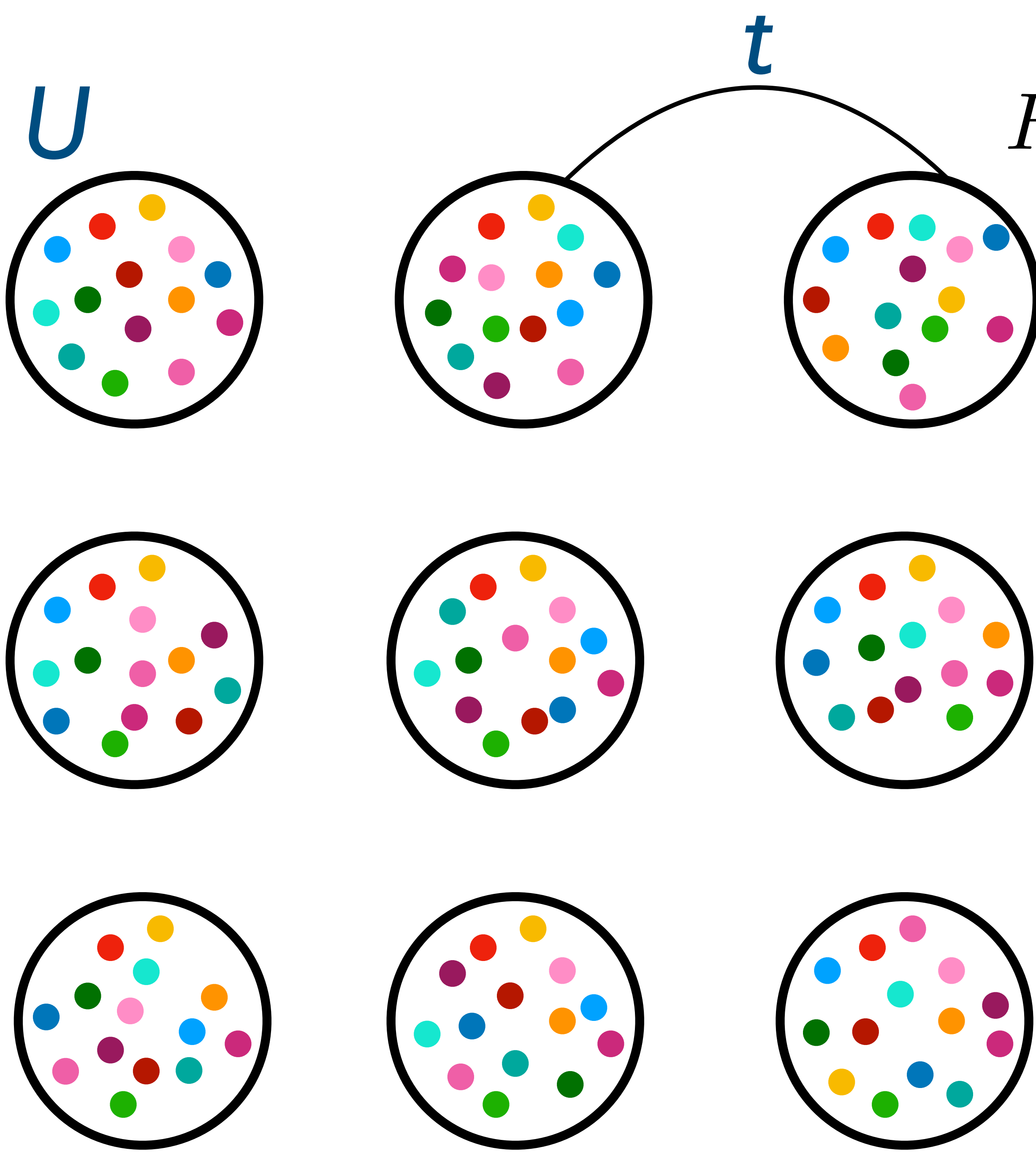
Figure: K. Fujita and J. C. Seamus Davis



S. Kasahara, T. Shibauchi, K. Hashimoto, K. Ikada, S. Tonegawa, R. Okazaki, H. Shishido,
H. Ikeda, H. Takeya, K. Hirata, T. Terashima, and Y. Matsuda,
Physical Review B **81**, 184519 (2010)

Coupled SYK Islands

SYK quantum islands of electrons with random hopping between them.



$$H = \sum_x \sum_{i<j,k<l} U_{ijkl,x} c^\dagger_{ix} c^\dagger_{jx} c_{kx} c_{lx}$$

$$+ \sum_{\langle xx' \rangle} \sum_{i,j} t_{ij,xx'} c^\dagger_{i,x} c_{j,x'}$$

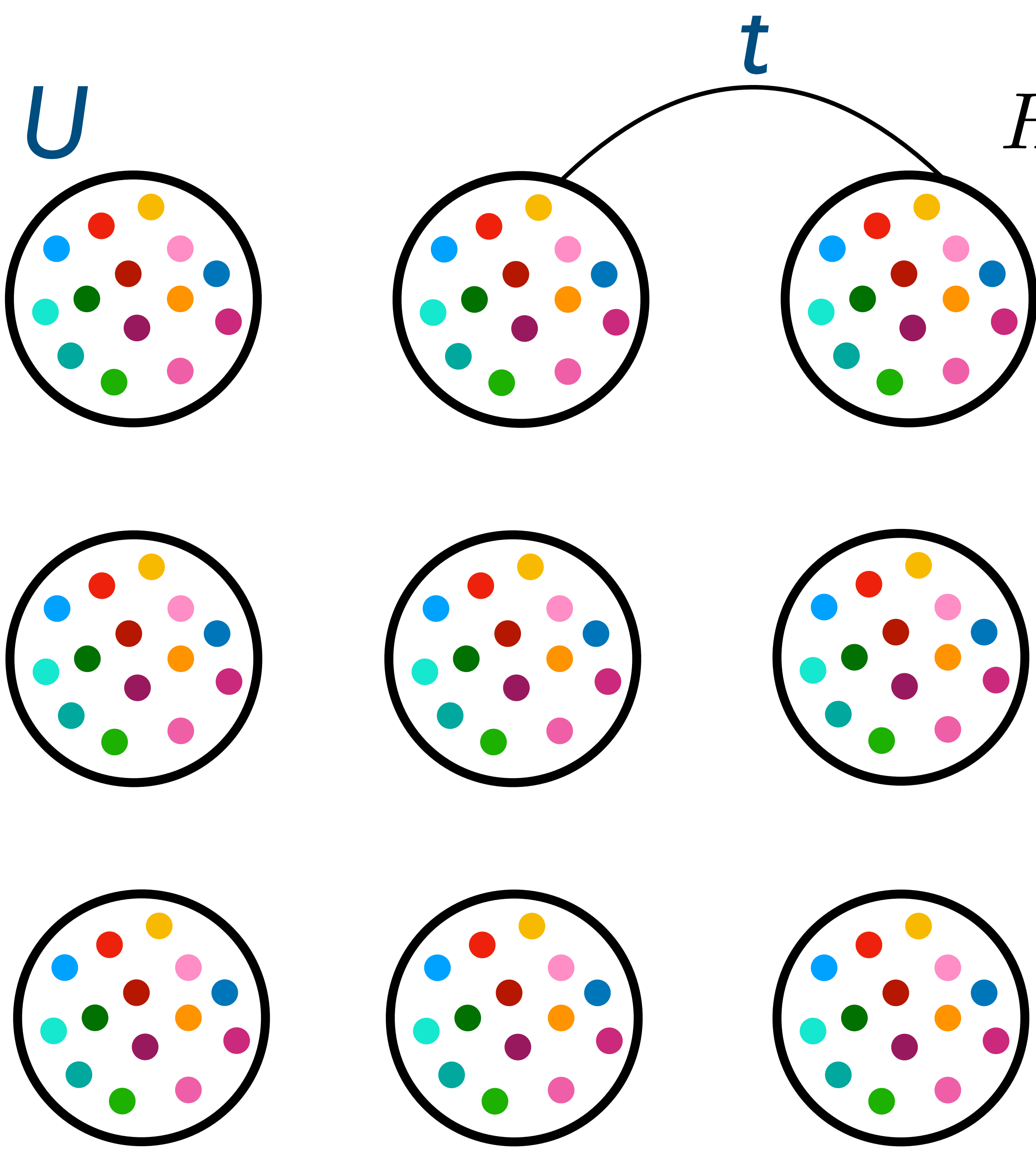
$$\overline{|U_{ijkl}|^2} = \frac{2U^2}{N^3} \qquad \overline{|t_{ij,x,x'}|^2} = t_0^2/N$$

Xue-Yang Song, Chao-Ming Jian, and L. Balents, PRL **119**, 216601 (2017)

See also A. Georges and O. Parcollet PRB **59**, 5341 (1999)

Coupled SYK Islands

Can also use non-random *t*, and the same *U* on all "islands".



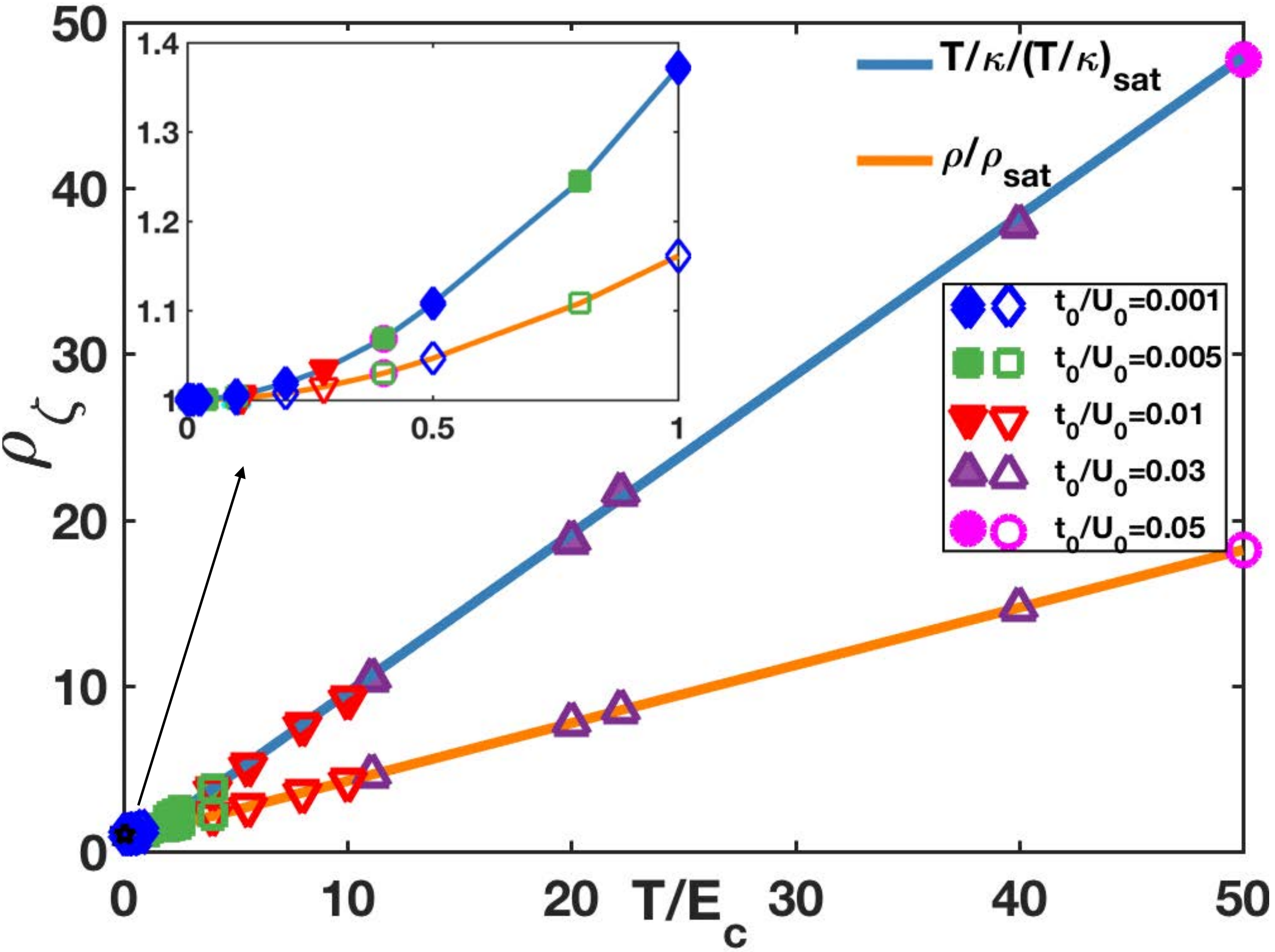
$$H = \sum_x \sum_{i<j,k<l} U_{ijkl} \quad c^\dagger_{ix} c^\dagger_{jx} c_{kx} c_{lx}$$

$$+ \sum_{\langle xx' \rangle} \sum_{i,j} t_{ij} \quad c^\dagger_{i,x} c_{j,x'}$$

Pengfei Zhang, PRB **96**, 205138 (2017)
Debanjan Chowdhury, Yochai Werman, Erez Berg, T. Senthil, arXiv:1801.06178
See also A. Georges and O. Parcollet PRB **59**, 5341 (1999)

Coupled SYK Islands

Low 'coherence' scale



$$E_c \sim \frac{t_0^2}{U}$$

Xue-Yang Song, Chao-Ming Jian, and L. Balents, PRL **119**, 216601 (2017)

See also A. Georges and O. Parcollet PRB **59**, 5341 (1999)

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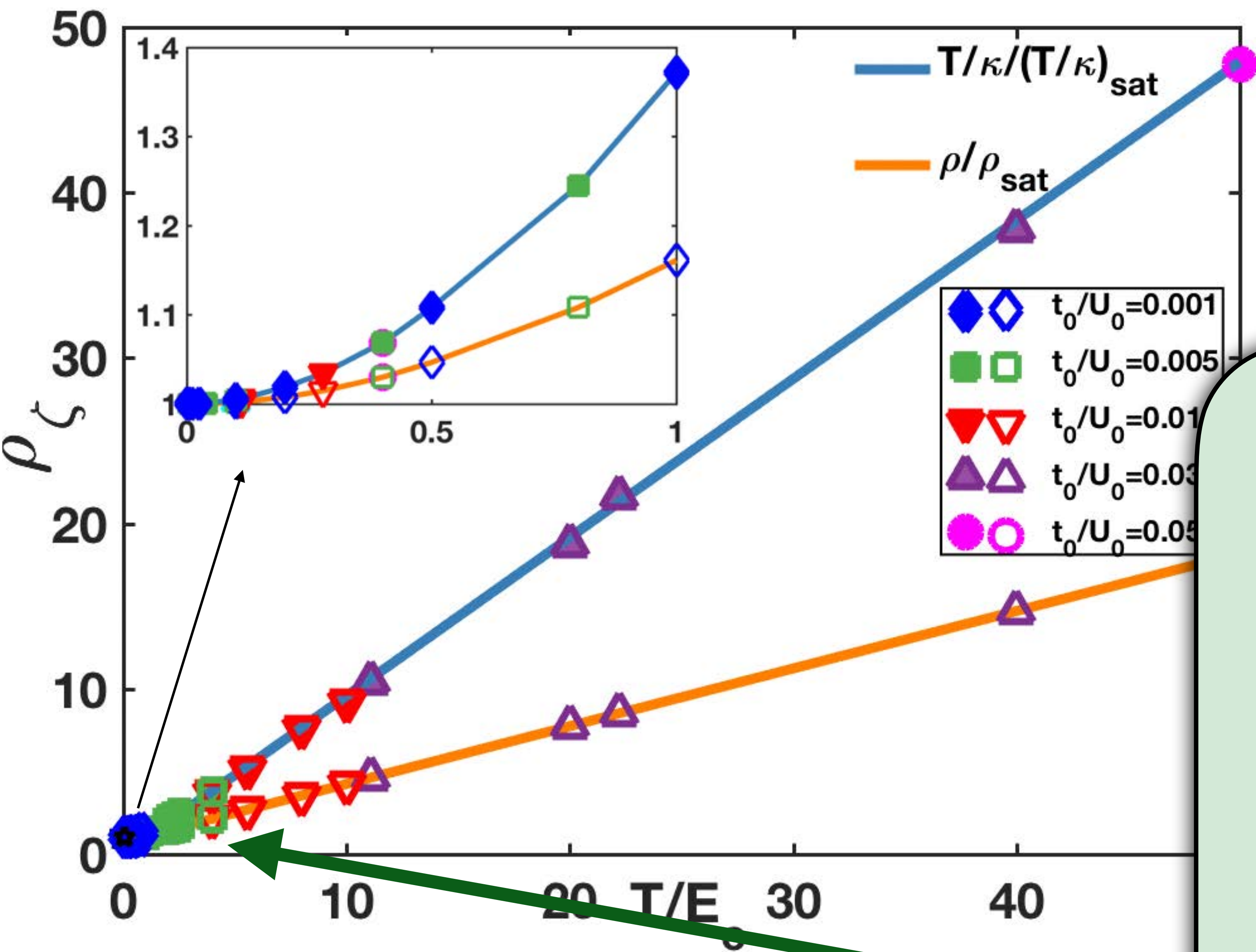


Low 'coherence' scale

$$E_c \sim \frac{t_0^2}{U}$$

For $T < E_c$, the resistivity, ρ, and entropy density, s, are

$$\rho = \frac{h}{e^2}\left[c_1 + c_2\left(\frac{T}{E_c}\right)^2\right]$$

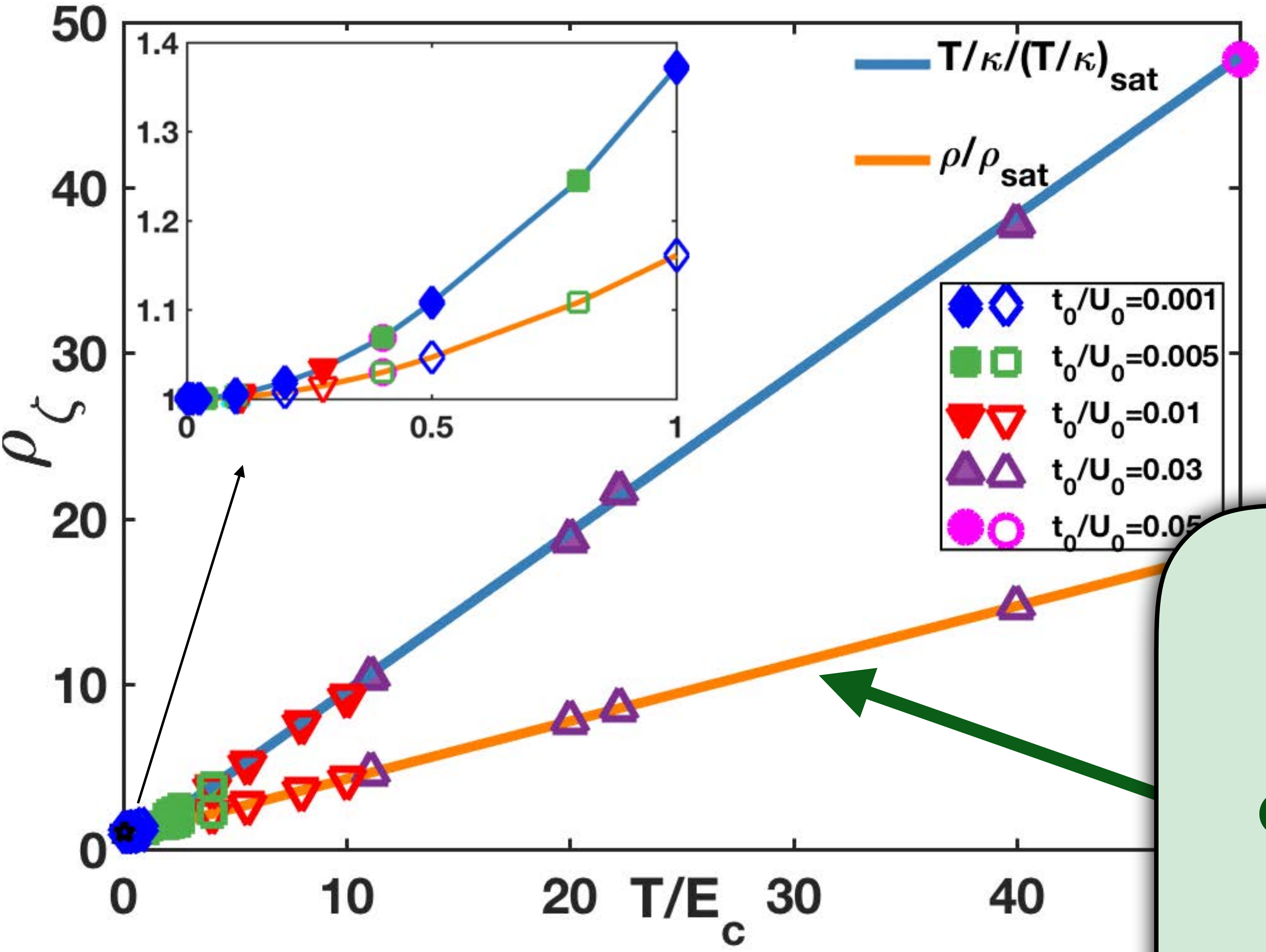
$$s \sim s_0\left(\frac{T}{E_c}\right)$$

Xue-Yang Song, Chao-Ming Jian, and L. Balents, PRL **119**, 216601 (2017)
See also A. Georges and O. Parcollet PRB **59**, 5341 (1999)

Coupled SYK Islands

Low 'coherence' scale

$$E_c \sim \frac{t_0^2}{U}$$



For $E_c < T < U$, the resistivity, ρ, and entropy density, s, are

$$\rho \sim \frac{h}{e^2}\left(\frac{T}{E_c}\right) \quad , \qquad s = s_0$$

Xue-Yang Song, Chao-Ming Jian, and L. Balents, PRL **119**, 216601 (2017)

See also A. Georges and O. Parcollet PRB **59**, 5341 (1999)

Quantum matter without quasiparticles

- Rapid local thermal equilibration (of fermion correlators) in a 'Planckian' time

$$\tau_{\rm eq} \sim \frac{\hbar}{k_B T} \quad , \quad \text{as } T \to 0.$$

- Presence of quasiparticles should slow down thermalization, so *all* quantum systems obey

S. Sachdev, *Quantum Phase Transitions*, Cambridge (1999)

$$\tau_{\rm eq} > C \frac{\hbar}{k_B T} \quad , \quad \text{as } T \to 0.$$

Absence of quasiparticles $\Leftrightarrow$ Fastest possible thermalization

Quantum matter without quasiparticles

- Planckian dynamics is realized in the 'solvable' SYK models
- Black holes thermalize in a time $\sim \hbar/(k_B T_H)$, where T_H is the Hawking temperature.
- A Schwarzian theory of a time reparameterization mode, with SL(2,R) symmetry, describes the quantum dynamics of
  - the SYK models
  - black holes with near-extremal AdS_2 horizons