# Recent Results and Challenges in Glassy and Out of Equilibrium Dynamics

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# Mean-Field Disordered Systems

Spin-glasses, glasses, optimization problems, neural networks,...

• Thermodynamics

From Parisi's solution of mean-field spin-glasses to rigorous proofs of replica symmetry breaking phase ('79-now)

• Dynamics (Langevin, Montecarlo)

From Cugliandolo-Kurchan solution of the out of equilibrium dynamics to a theory of aging in glassy dynamics ('93-now)

Open problems and New Questions on Dynamics

- Dynamics due to non-conservative forces and chaos
- Barrier-crossing and dynamics on exponentially large time-scales

# Lotka-Volterra equations for large interacting ecosystems

$$\frac{dN_i}{dt} = N_i \left[ (1 - N_i) - \sum_{j,(j \neq i)} \alpha_{ij} N_j \right] + \lambda \qquad N_i \geq 0 \qquad i = 1, \ldots, S$$

$$S \to \infty$$

$\alpha_{ij}$ non-symmetric Gaussian random matrix with i.i.d elements

$$\langle \alpha_{ij} \rangle = \frac{\mu}{S} \qquad \langle \alpha_{ij}^2 \rangle_c = \frac{\sigma^2}{S} \qquad \langle \alpha_{ij} \alpha_{ji} \rangle_c = \gamma \langle \alpha_{ij}^2 \rangle_c \qquad -1 \leq \gamma \leq 1$$

- Dynamical phase diagram and dynamical phases?
- Out of equilibrium dynamics?
- Transition to chaos?
- Relationship with spin-glasses?

- High-dimensional disordered dynamical system
- A representative model to address new and central issues in ecology-biology (large ecosystems e.g. Human Microbiome)

# Dynamical Mean Field Theory

One species usual part

Direct effect of interactions

Feedback loop from interactions

Auxiliary field to define the response

$$\dot{N} = N\{1 - N - \mu m(t) - \sigma\eta(t) - \gamma\sigma^2 \int_0^t K(t,s)N(s)ds - h(t)\}$$

- the **average** $m(t)$
- the **correlator** $C(t,t') = \langle \eta(t)\eta(t')\rangle$
- the **averaged response function** $K(t,s)$

Similar to DMFT for spin-glasses: Sompolinsky, Zippelius (1981); Ben Arous, Dembo, Guionnet (2000)

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- the **average** $m(t) = \langle N(t) \rangle$;
- the **correlator** $C(t,t') = \langle \eta(t)\eta(t') \rangle = \langle N(t)N(t') \rangle$;
- the **averaged response function** $K(t,s) = \langle \frac{dN(t)}{dh(s)} \rangle_{h=0}$

Solution → $\frac{1}{S}\sum_i \langle N_i(t) \rangle = m(t)$ $\quad$ $\frac{1}{S}\sum_i \langle N_i(t)N_i(t') \rangle = C(t,t')$

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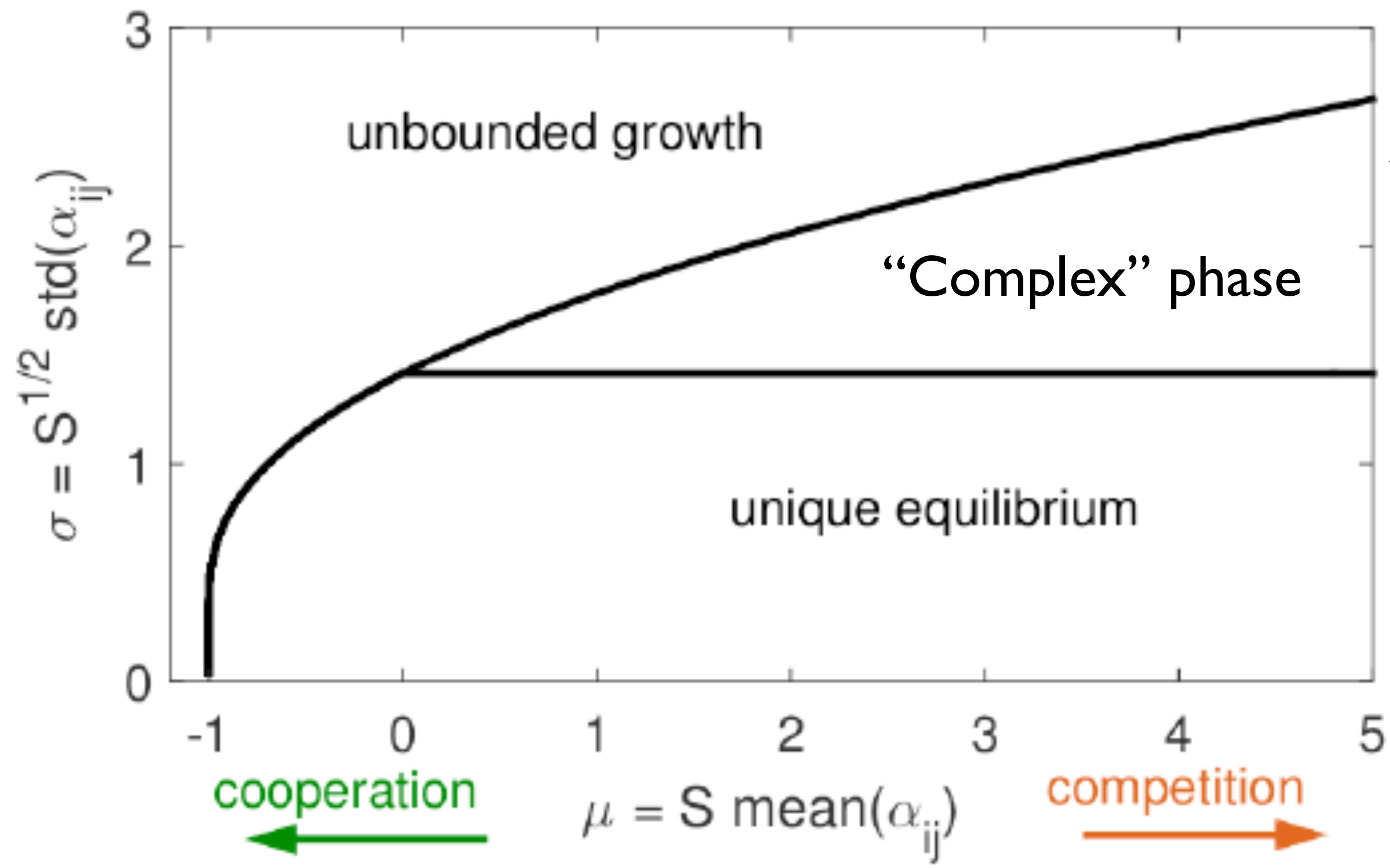
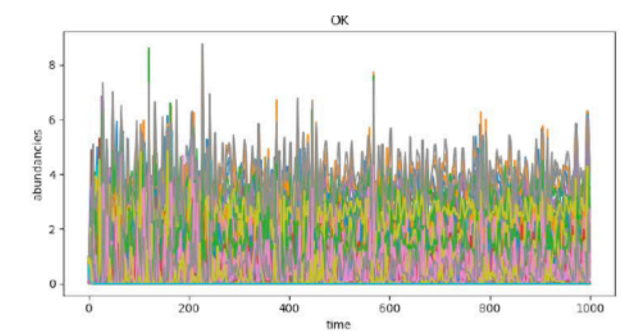
# Dynamical Phase Diagram

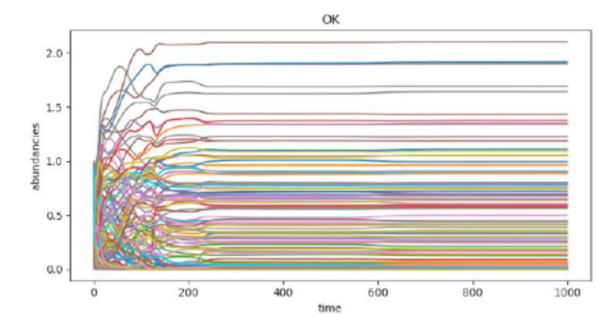



$\gamma = 1$ aging like in a spin-glass

$\gamma < 1$

Chaos




**EXACT SOLUTION**

**G. B., G. Bunin and C. Cammarota arXiv:1710.03606**
**and works in progress (F. Roy, V. Ros)**

Related works and phase diagrams
Sompolinsky, Crisanti, Sommers '88 ; Diederich, Opper '89;
Fisher, Mehta '14; Kessler, Shnerb '15; Bunin '16

- Properties of the transition to chaos?
- Stable chaos without immigration?
- Other equations, new phases?

# Barriers and Dynamics on Exponentially Long Time-Scales

Stochastic Dynamics of Mean-Field Glassy Models (e.g. p-spin spherical model)

$$E = -\sum_{\langle i_1,\cdots,i_p\rangle} J_{i_1,\cdots,i_p} S_{i_1}\cdots S_{i_p} \qquad \sum_{i_1=1}^{N} S_{i_1}^2 = N$$

$$i_1 = 1,\cdots,N$$

## Very Rough Energy Landscape

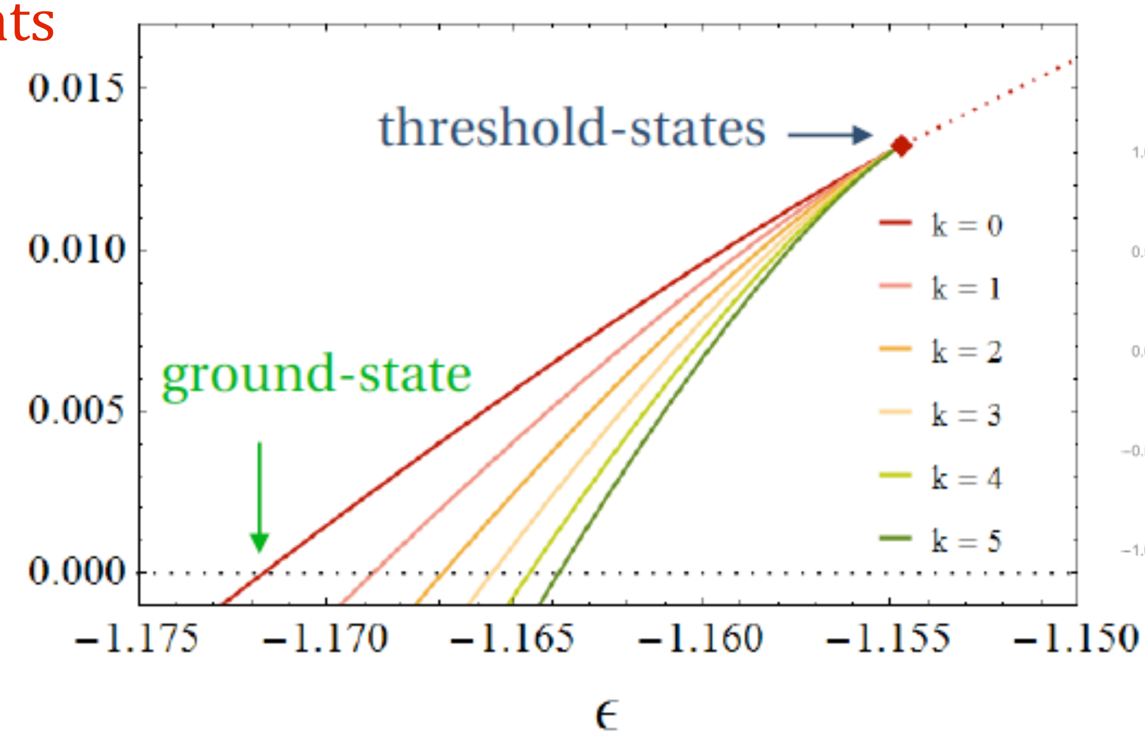
Exponential number of critical points

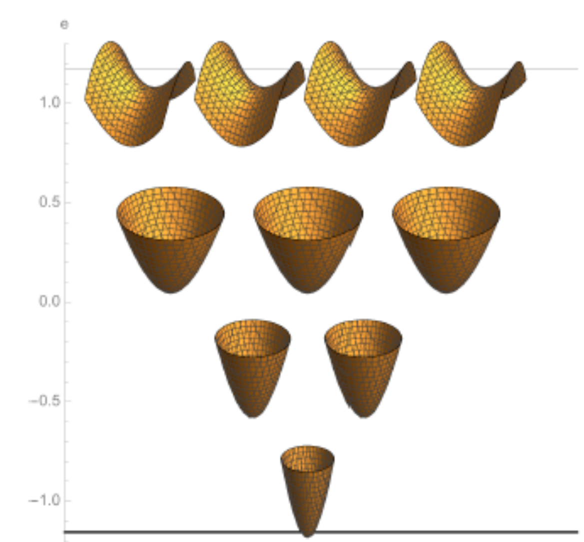
$$\mathcal{N}(e) \sim e^{N\Sigma_k(e)}$$

$\Sigma_k(e)$ Complexity of critical points of index k

Cavagna, Giardina, Parisi (1998)
Auffinger, Ben Arous, Cerny (2013)

$$\Sigma_k(e) = \frac{1}{N}\ln\mathcal{N}(e)$$




barriers ~ N

•T<Td out of equilibrium dynamics: aging for $t\to\infty$ after $N\to\infty$ (no barrier crossing)

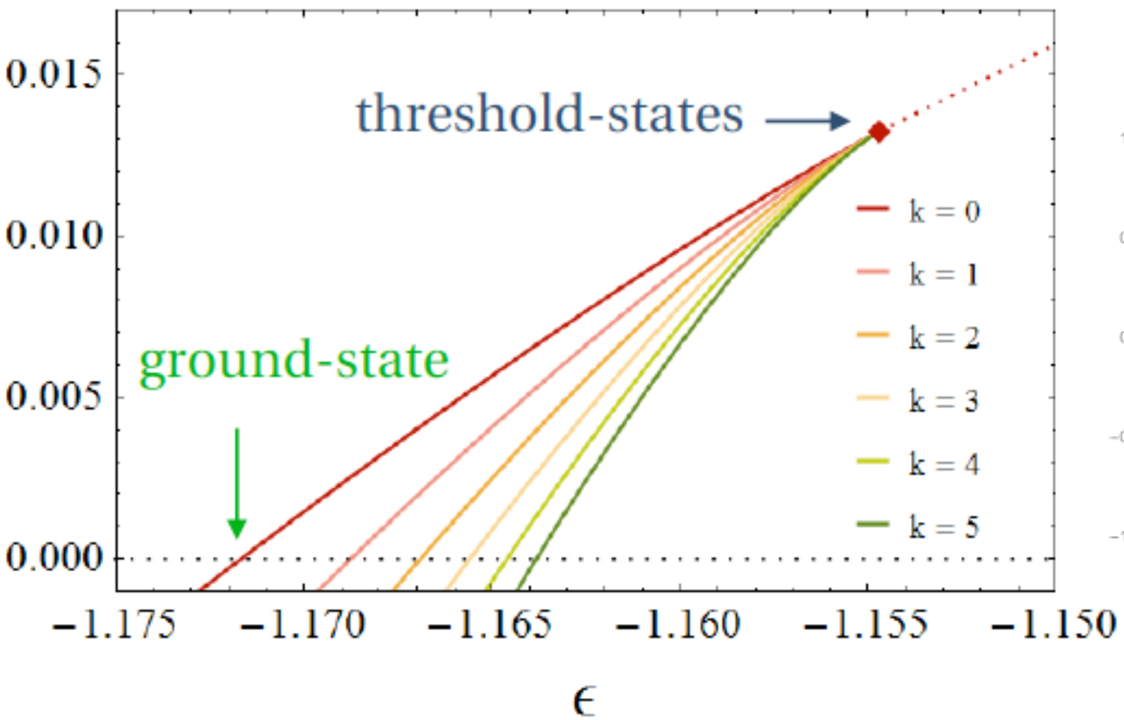
Dynamics on exponentially large in N time-scales?

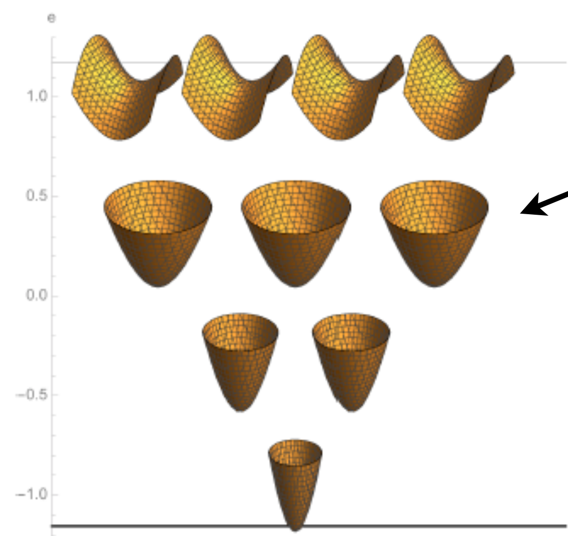
Activated dynamics and Barrier Crossing?

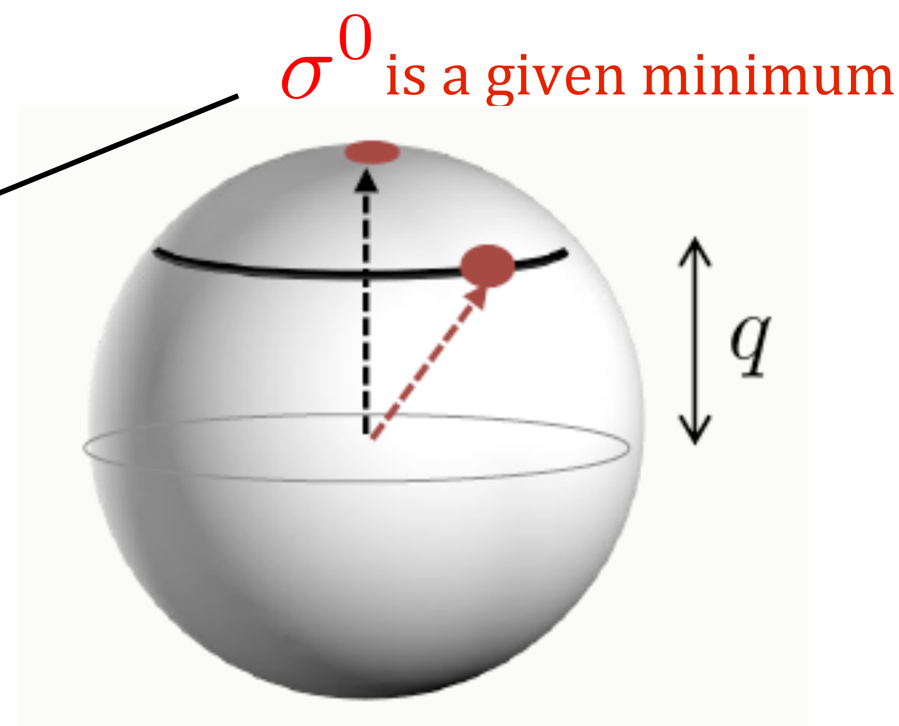
# Complexity of Barriers

•Using the Kac-Rice method developed in V.Ros, G. Ben Arous, G. Biroli, C. Cammarota, 2018

$$\Sigma(\epsilon, q|\epsilon_0) = \lim_{N\to\infty} \frac{1}{N} \left\langle \log \mathcal{N}_{\boldsymbol{\sigma}^0}(\epsilon, q|\epsilon_0) \right\rangle$$






V. Ros, G. Biroli, C. Cammarota to appear

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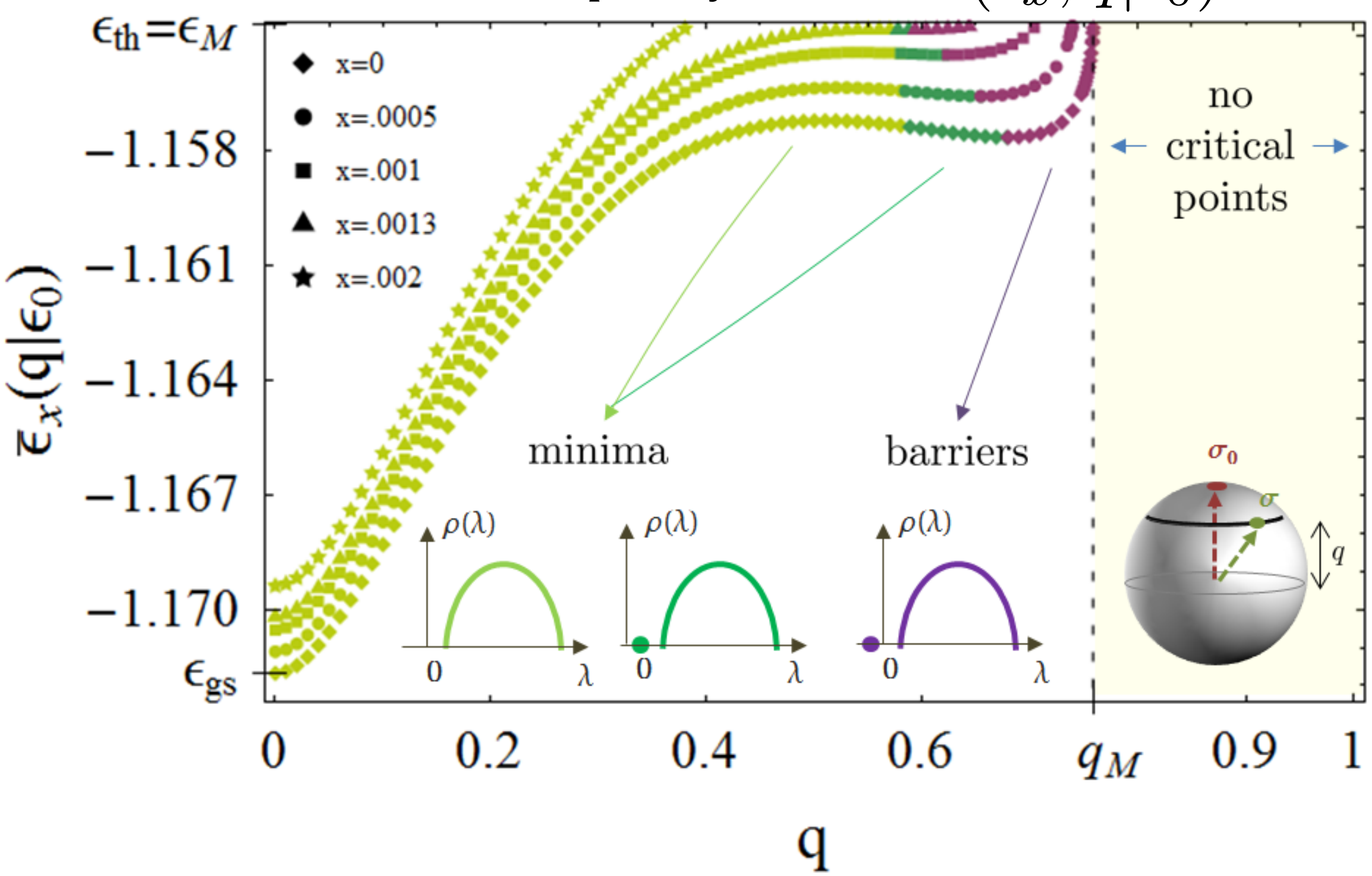
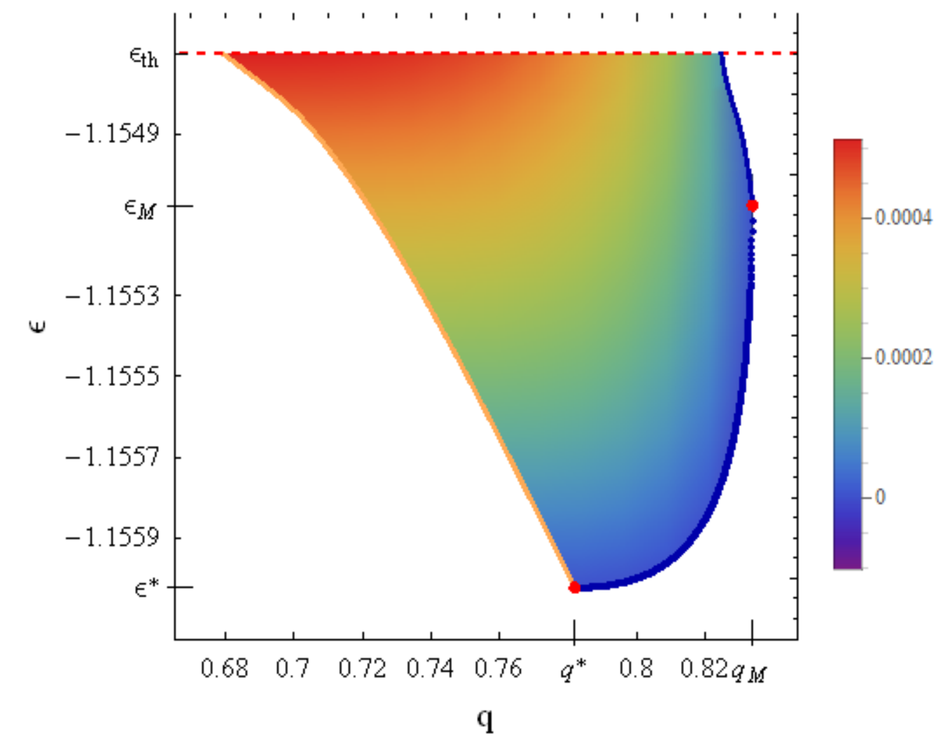
Iso-complexity curves: $\Sigma(\overline{\epsilon}_x, q|\epsilon_0) = x$




- Lowest barrier to escape from a minimum is lower than the threshold

- Exponential number of barriers

Full geometrical structure of barriers?
Generalized Freidlin-Wentzell theory?

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# Dynamics

•Full analysis in the Random Energy Model: on exponentially large time-scales the aging dynamics is the one of the Bouchaud trap model

Simple arguments, simulations and rigorous proofs: Ben Arous, Bovier, Gayrard (2002), Junier, Kurchan (2004), Cerny, Wassmer (2017), Gayrard (2018), Baity-Jesi, Biroli, Cammarota (2018)

For general mean-field glassy systems: a central open problem

Promising route: large deviation theory combined with dynamical mean-field theory and inspired by the results on complexity of barriers