Measure of Maximal Entropy for the Finite Horizon Periodic Lorentz Gas

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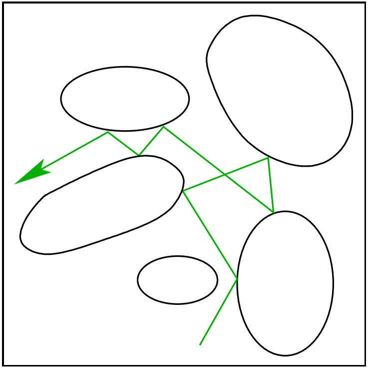
joint work with Viviane Baladi

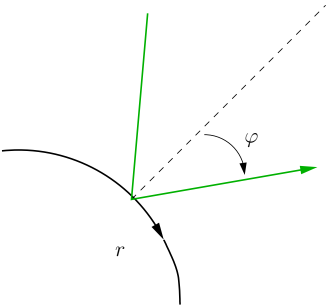
Finite Horizon Sinai Billiard

- Billiard table $\mathcal{Q} = \mathbb{T}^2 \backslash \cup_i B_i$; scatterers B_i.
- Boundaries of scatterers are $\mathcal{C}^3$ and have strictly positive curvature.
- Billiard flow is given by a point particle moving at unit speed with elastic collisions at the boundary

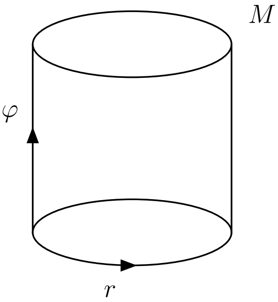


Assume a **finite horizon** condition: there is an upper bound on the free flight time between consecutive tangential collisions on the table.

The Associated Billiard Map



- $r =$ position coordinate oriented clockwise on boundary of scatterer ∂B_i
- $\varphi =$ angle outgoing trajectory makes with normal to scatterer



$M = \big(\cup_i \partial B_i \big) \times [-\frac{\pi}{2}, \frac{\pi}{2}]$, the natural "collision" cross-section for the billiard flow.

$T : (r, \varphi) \to (r', \varphi')$ is the first return map: the billiard map.

- a hyperbolic map with singularities

Statistical Properties

T preserves a smooth invariant measure on M, $\mu_{\mathsf{SRB}} = \cos\varphi \, dr \, d\varphi$

With respect to this measure, many statistical properties have been proved using a variety of techniques.

With respect to μ_{SRB}, T:

- is ergodic [Sinai '70] and Bernoulli [Gallavotti, Ornstein '74]
- enjoys exponential decay of correlations [L.S. Young '98]
- satisfies many limit theorems:
  - Central Limit Theorem [Bunimovich, Sinai '81]
  - Almost-sure invariance principle [Melbourne, Nicol '05],
  - Local moderate and large deviations [Melbourne, Nicol '08], [Young, Rey-Bellet '08]

Very few results on existence of other invariant measures for T: [Chen, Wang, Zhang '17] treats 'perturbations' of μ_{SRB}.

Topological Entropy of Billiard Map

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- Indeed, even a definition for topological entropy is not straightforward, and can depend on the choice of metric.

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Known Results:

- Tangential collisions: $\mathcal{S}_0 = \{(r, \varphi) \in M : \varphi = \pm\frac{\pi}{2}\}$
- For $n \in \mathbb{Z}$, $\mathcal{S}_n = \cup_{i=0}^{n} T^{-i}\mathcal{S}_0$ is the singularity set for T^n.
- Define $M' = M \setminus (\cup_{n=-\infty}^{\infty} \mathcal{S}_n)$. $T : M' \circlearrowleft$ is a continuous map.

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[Chernov '91] studied the topological entropy of T on an invariant subset $M_1 \subset M'$ using a countable Markov partition. He showed

$$h_{\text{top}}(T, M') \geq h_{\text{top}}(T, M_1) = h_{\text{top}}(\sigma, \Sigma_1),$$

where (σ, Σ_1) is the TMC derived from the Markov partition.

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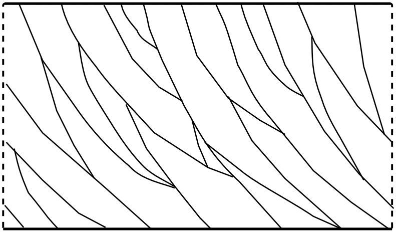
Once Step 2 is carried out, one can ask about properties of the measure μ_*: Is it ergodic, mixing, Bernoulli?

Step 1: A Definition of Topological Entropy

- Let $\mathcal{M}^n_{-k}$ = connected components of $M \setminus (\mathcal{S}_{-k} \cup \mathcal{S}_n)$.
- Define

$$h_* = \lim_{n\to\infty} \frac{1}{n} \log \#\mathcal{M}_0^n$$



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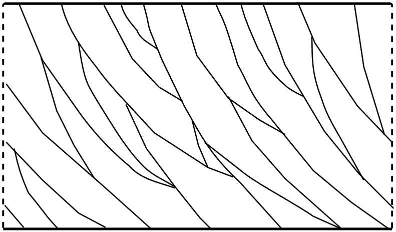

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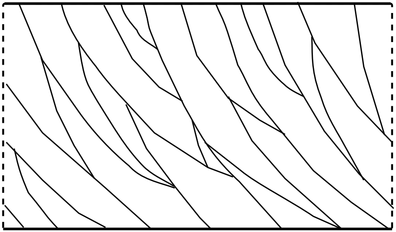

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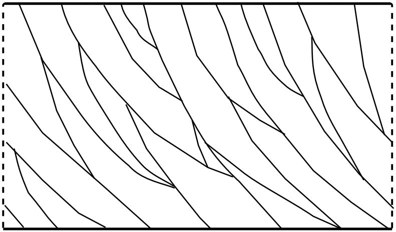

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- The limit exists since the sequence $\log \#\mathcal{M}_0^n$ is subadditive: $\#\mathcal{M}_0^{n+m} \le \#\mathcal{M}_0^n \cdot \#\mathcal{M}_0^m$.
- h_* is the exponential rate of growth of the number of pieces created by the discontinuities of T. It does not depend on a choice of metric.
- $T^n \mathcal{S}_n = \mathcal{S}_{-n} \implies \#\mathcal{M}_0^n = \#\mathcal{M}_{-n}^0$. So $h_*(T) = h_*(T^{-1})$.

Dynamical Refinements of Partitions

In order to connect h_* to measure theoretic entropy, it is convenient to express it in terms of dynamical refinements of partitions. Define

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- $\mathring{\mathcal{P}}_{-k}^{n} = \mathcal{M}_{-k-1}^{n+1}$
- $\#\mathcal{P}_{-k}^{n} \leq \#\mathring{\mathcal{P}}_{-k}^{n} + C(k+n+1)$, C depends only on the table

Characterization of h_* and Variational Inequality

$$h_* = \lim_{n\to\infty} \frac{1}{n} \log \#\mathcal{M}_0^n$$

Theorem 1

For a finite horizon Lorentz gas,

- $h_* = h_{\mathsf{sep}} = h_{\mathsf{span}}$
- For any $k \geq 0$,

$$h_* = \lim_{n\to\infty} \frac{1}{n} \log \#\mathcal{P}_{-k}^n = \lim_{n\to\infty} \frac{1}{n} \log \#\mathring{\mathcal{P}}_{-k}^n$$

- h_* satisfies a variational inequality,

$$h_* \geq \sup\{h_\mu(T) : \mu \text{ is a } T\text{-invariant Borel prob. measure}\}$$

Step 2: Construction of a Measure of Maximal Entropy

We introduce an additional assumption on T.

- Fix $n_0 \in \mathbb{N}$ and an angle φ_0 close to $\pi/2$.
- Let $s_0 \in (0,1)$ be the smallest number such that any orbit of length n_0 has at most $s_0 n_0$ collisions with $|\varphi| \geq \varphi_0$.

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Assumption: $h_* > s_0 \log 2$

If W is a local stable manifold, then $|T^{-1}W| \leq C|W|^{1/2}$.

Our assumption ensures that the growth due to tangential collisions does not exceed the exponential rate of growth given by h_*.

Weighted Transfer Operator

For a smooth test function ψ, define a weighted transfer operator $\mathcal{L}$ acting on a distribution μ on M by

$$\mathcal{L}\mu(\psi) = \mu\left(\frac{\psi \circ T}{J^sT}\right), \quad \psi \text{ a test function,}$$

where $J^sT \approx \cos\varphi$ denotes the stable Jacobian of T.

Recall that T preserves a smooth invariant measure
$d\mu_{\mathsf{SRB}} = c\cos\varphi\, dr d\varphi$.
If $d\mu = f d\mu_{\mathsf{SRB}}$ is a measure abs. cont. w.r.t. μ_{SRB}, then

$$\mathcal{L}f(x) = \frac{f(T^{-1}x)}{J^sT(T^{-1}x)}.$$

We want to construct a measure of maximal entropy out of left and right eigenvectors of this operator.

Norms for the Operator

Norms similar to those used in [D., Zhang '11] for the transfer operator with respect to the SRB measure, but several differences in order to compensate for the potential $1/J^sT$, which blows up near tangential collisions.

- Norms integrate on real (local) stable manifolds, $\mathcal{W}^s$, rather than admissible cone-stable curves.
- We do not subdivide curves according to homogeneity strips.
- The test functions have a logarithmic, rather than Hölder, weight on the size of the curve. This is a crucial change:
  - We need it to compensate for the fact that for $W \in \mathcal{W}^s$, $|T^{-1}W|$ can be of order $|W|^{1/2}$, yet the weight in the transfer operator cancels the Jacobian that would help us in this case.
  - It prevents us from proving true Lasota-Yorke inequalities: $\mathcal{L}$ is not quasi-compact!

Banach Spaces

Theorem 2

- We have a sequence of inclusions,

$$\mathcal{C}^1(M) \subset \mathcal{B} \subset \mathcal{B}_w \subset (\mathcal{C}^\alpha(M))^*.$$

- The embedding of the unit ball of $\mathcal{B}$ in $\mathcal{B}_w$ is compact.
- The following inequalities hold: There exists $C > 0$ such that for all $f \in \mathcal{B}$, $n \geq 0$,

$$|\mathcal{L}^n f|_w \leq C|f|_w \# \mathcal{M}_0^n\,,$$
$$\|\mathcal{L}^n f\|_s \leq C\|f\|_s \# \mathcal{M}_0^n\,,$$
$$\|\mathcal{L}^n f\|_u \leq C(\|f\|_u + \|f\|_s) \# \mathcal{M}_0^n\,.$$

The inequalities above are not true Lasota-Yorke inequalities due to lack of contraction in the strong norm.

Exact Exponential Growth of $\mathcal{M}_0^n$

To obtain a precise estimate on the spectral radius of $\mathcal{L}$, we prove the following inequalities: There exists $c_1 > 0$ such that,

$$e^{nh_*} \le \#\mathcal{M}_0^n \le c_1 e^{nh_*} \quad \text{for all } n \ge 1.$$

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This, in turn, relies on several growth/fragmentation lemmas.

Lemma (Growth Lemma)

- For a local stable manifold $W \in \mathcal{W}^s$, most pieces of $T^{-n}W$ are longer than some length scale δ.
- Most components of $\mathcal{M}_0^n$ have stable diameter longer than δ. Similarly, most components of $\mathcal{M}_{-n}^0$ have unstable diameter longer than δ.

- These are distinct from the usual growth lemmas since there are no homogeneity strips and no Jacobian appears as a weight in the sum.

Construction of μ_*

- The sequence

$$\nu_n = \frac{1}{n}\sum_{k=0}^{n-1} e^{-kh_*}\mathcal{L}^k 1, \quad \text{is uniformly bounded in } \mathcal{B}.$$

By compactness, a subsequence converges in $\mathcal{B}_w$.
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Since $\mathcal{L}\nu = e^{h_*}\nu$ and $\mathcal{L}^*\tilde{\nu} = e^{h_*}\tilde{\nu}$, we have $\mu_*(\psi \circ T) = \mu_*(\psi)$, i.e. μ_* is an invariant measure for T.

Hyperbolicity of μ_*

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This implies estimates of the form:

- For any $k \in \mathbb{Z}$, $\exists C_k > 0$ s.t.

$$\nu(\mathcal{N}_\varepsilon(\mathcal{S}_k)) \le C_k(-\log \varepsilon)^{-\gamma}, \qquad \mu_*(\mathcal{N}_\varepsilon(\mathcal{S}_k)) \le C_k(-\log \varepsilon)^{-\gamma}.$$

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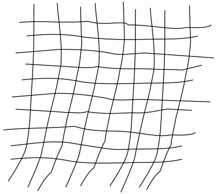
- μ_*-a.e. $x \in M$ has a stable and unstable manifold of positive length. The same is true with respect to ν.

Ergodicity of μ_*

Since μ_* is hyperbolic, we cover a full measure set of M with Cantor rectangles, and study the properties of μ_* on each rectangle.



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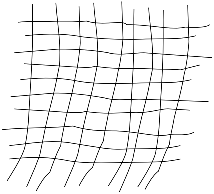

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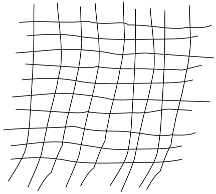

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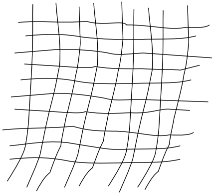

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- Each Cantor rectangle R belongs to one ergodic component of μ_*.
- Since T is topologically mixing T, we can force images of rectangles to overlap $\implies (T^n, \mu_*)$ is ergodic for all n.

Mixing and Bernoulli Property of μ_*

- The local product structure of the Cantor rectangles, together with a global argument showing that a full measure set of points on each component of M can be connected by a network of stable/unstable manifolds, enables us to prove that (T, μ_*) is K-mixing, following techniques of [Pesin '77, '92].

Mixing and Bernoulli Property of μ_*

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- K-mixing + hyperbolicity + absolute continuity of μ_* + bounds on $\mu_*(\mathcal{N}_\varepsilon(\mathcal{S}_{\pm 1}))$
  $\implies$ the partition $\mathcal{M}_{-1}^1$ is very weakly Bernoulli, following the technique of [Chernov, Haskell '96].

  Since $\bigvee_{n=-\infty}^{\infty} T^{-n}(\mathcal{M}_{-1}^1)$ generates the full σ-algebra for T, this implies by [Ornstein, Weiss '73] that (T, μ_*) is Bernoulli.

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- But $h_* \ge h_{\mu_*}(T)$ by Theorem 1.
- Conclude: $h_* = h_{\mu_*}(T)$.

Variational Principle and Measure of Maximal Entropy

Theorem 3

Let T be the billiard map corresponding to a finite horizon periodic Lorentz gas. Assume $h_* > s_0 \log 2$. Then,

$$h_* = \lim_{n\to\infty} \frac{1}{n} \log \#\mathcal{M}_0^n = \sup_{\mu} h_\mu(T).$$

Moreover, there exists a T-invariant measure μ_* such that

- $h_{\mu_*}(T) = h_*$
- $h_* = h_{\text{top}}(T, M')$
- (T, μ_*) is Bernoulli and positive on open sets
- $\int -\log d(x, \mathcal{S}_{\pm 1})\, d\mu_*(x) < \infty$