Singularities of dynamic response functions in the massless regime of the XXZ chain

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Outline

The XXZ chain

⊛ The XXZ spin-1/2 chain on $\mathfrak{h}_{XXZ} = \otimes_{n=1}^{L} \mathbb{C}^2$, σ^α Pauli matrices

$$\mathrm{H} = \sum_{n=1}^{L} \left\{ \sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \Delta \sigma_n^z \sigma_{n+1}^z - h \sigma_n^z \right\} \quad , \quad \sigma_{n+L}^\alpha \equiv \sigma_n^\alpha$$

Local operators : $\sigma_n^\alpha = \underbrace{\mathrm{id} \otimes \cdots \otimes \mathrm{id}}_{n-1 \text{ times}} \otimes \sigma^\alpha \otimes \underbrace{\mathrm{id} \otimes \cdots \otimes \mathrm{id}}_{L-n \text{ times}}$

⊛ Model solvable by Bethe Ansatz **'31** (Bethe), **'58** (Orbach): $\mathrm{H} \cdot \Upsilon = \widehat{\mathcal{E}}_\Upsilon \cdot \Upsilon$

✔ Ground state, Completeness, Norms, Scalar products, Matrix elements of local operators . . .

Yang, Yang, Gaudin, McCoy, Wu, Korepin, Slavnov, Kitanine, Maillet, Terras, Mukhin, Tarasov, Varchenko

⊛ Spectrum above ground state $\lim_{L \to +\infty} \{ \widehat{\mathcal{E}}_\Upsilon - \widehat{\mathcal{E}}_\Omega \}$

Hulten, Yang, Yang, Des Cloiseau, Gaudin, Ishumira, Shiba, Faddeev, Takhtadjan, Dorlas, Samsonov, Gusev, K.

⊛ Closed expression for static correlators

Izergin, Korepin, Jimbo, Miwa, Miki, Nakayashiki, Kitanine, Maillet, Slavnov, Terras

Dynamic response functions in the XXZ chain

- Time evolution of operators

$$\sigma_m^\gamma(t) = \mathrm{e}^{\mathrm{i}\mathrm{H}t} \cdot \sigma_m^\gamma \cdot \mathrm{e}^{-\mathrm{i}t\mathrm{H}}$$

- Connected dynamical two-point function at zero temperature

$$\Big\langle \big(\sigma_1^\gamma(t)\big)^\dagger \sigma_{m+1}^\gamma \Big\rangle_{\mathrm{c}} = \lim_{L\to+\infty} \Big\{ \Big(\Omega, \big(\sigma_1^\gamma(t)\big)^\dagger \sigma_{m+1}^\gamma \Omega\Big) - \Big|\big(\Omega, \sigma_1^\gamma \Omega\big)\Big|^2 \Big\}$$

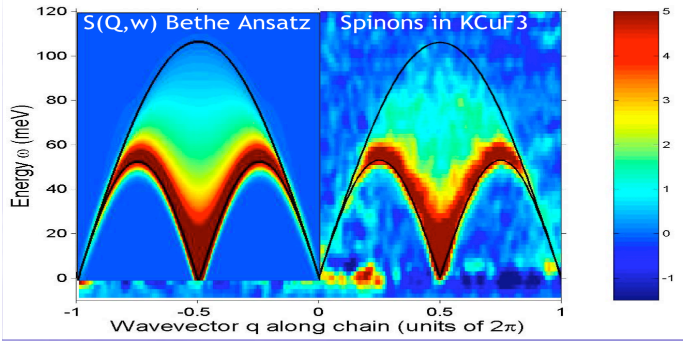
'95 Jimbo, Miwa , **'04** Kitanine, Maillet, Slavnov, Terras : Series representation.

- Experiments ⇝ Dynamic Response Functions

$$\mathscr{S}^{(\gamma)}(k,\omega) = \sum_{m\in\mathbb{Z}} \int_{\mathbb{R}} \Big\langle \big(\sigma_1^\gamma(t)\big)^\dagger \sigma_{m+1}^\gamma \Big\rangle_{\mathrm{c}} \cdot \mathrm{e}^{\mathrm{i}(\omega t - km)} \frac{\mathrm{d}t}{(2\pi)^2}$$

Expermiental measurements on $KCuF_3$

- Bragg spectroscopy.
- $KCuF_3 \quad \rightsquigarrow \quad$ XXX chain at $h = 0$.



'05 (Caux, Maillet, Hagemans) **'95** (Tennant, Cowley, Nagler, Tsvelik)

Observations for $\mathscr{S}^{(z)}(k,\omega)$

- Dominant intensity delimited by viking helmet curves
- Large intensity concentrated on lower curves
- Decrease of intensity when approaching top curve

State of the art for the response functions

♦ **Non-linear luttinger liquid phenomenology**

Mahan, Noziére, De Dominicus, Glazman, Kamenev, Khodas, Pustilnik, Imambekov, Cheianov, Affleck, Pereira, White, Caux, Shashi

$$\mathscr{S}^{(z)}(k,\omega) \simeq \mathscr{A}(k) \cdot \Xi\,(\omega - \varepsilon_h(k)) \cdot [\omega - \varepsilon_h(k)]^{\vartheta}$$

- Universal structure of the amplitudes

$$\mathscr{A}(k) = \frac{(2\pi)^2\,\Gamma^{-1}(\mu_R + \mu_L)}{[v(k)+v_F]^{\mu_R}\,[v_F - v(k)]^{\mu_L}} \cdot \left|\mathcal{F}^{(z)}(k)\right|^2 \quad , \quad \vartheta = \mu_R + \mu_L - 1$$

- Bethe Ansatz spectrum $\rightsquigarrow$ closed expressions for $\mu_{R,L}$;
- $\left|\mathcal{F}^{(z)}(k)\right|^2 \rightsquigarrow$ volume-renormalised form factor;

♦ **Integrability based investigation**

⊛ $\mathscr{S}^{(z)}(k,\omega) = \sum_{n\geq 0} \mathcal{S}_{2n}(k,\omega)$ for XXX $h=0$ & (k,ω) behaviour from $n=1,2$
Jimbo, Miwa, Bougourzi, Couture, Kacir, Karbach, Müller, Mütter, Caux, Konno, Sorrel, Weston

⊛ Numerics & Bethe Ansatz
Biegel, Karbach, Müller, Sato, Shiroishi, Takahashi, Caux, Hagemans, Maillet,...

⊛ Formal saddle-point approach to form factor series in NLSM
Kitanine, K., Maillet, Slavnov, Terras

The form factor expansion of dynamic response functions

⊛ ['**17** K.] Summation over all sectors of excitations

$$\mathscr{S}^{(\gamma)}(k,\omega) = \sum_{\boldsymbol{n}\in\mathfrak{S}} \mathscr{S}^{(\gamma)}_{\boldsymbol{n}}(k,\omega)$$

$\mathscr{S}^{(\gamma)}_{\boldsymbol{n}}(k,\omega)$: muti-dimensional integral

Questions

- Can one develop a method to study the behaviour of $\mathscr{S}^{(\gamma)}_{\boldsymbol{n}}(k,\omega)$ in (k,ω)?
- Can one access to the predicted universal features for an integrable model?
- Is the non-linear Luttinger picture complete?

The model integral

$$\mathcal{I}(\mathfrak{x}) = \prod_{r=1}^{\ell}\Big\{\int_{\mathscr{I}_r^{n_r}} \mathrm{d}\boldsymbol{p}^{(r)} \prod_{a<b}^{n_r}\big(p_a^{(r)}-p_b^{(r)}\big)^2\Big\}\cdot \mathscr{G}\big(\boldsymbol{p},\mathfrak{z}_+(\boldsymbol{p};\mathfrak{x}),\mathfrak{z}_-(\boldsymbol{p};\mathfrak{x})\big)\cdot\prod_{\upsilon=\pm}\Big\{\Xi\big(\mathfrak{z}_\upsilon(\boldsymbol{p};\mathfrak{x})\big)\cdot\big[\mathfrak{z}_\upsilon(\boldsymbol{p};\mathfrak{x})\big]^{\Delta_\upsilon(\boldsymbol{p})-1}\Big\}$$

- $\mathscr{G}\big(\boldsymbol{p},x,y\big) = G(\boldsymbol{p}) + \mathrm{O}\big(x^\alpha+y^\alpha\big)$ $\qquad \mathscr{G}=0$ on $\partial\Big\{\prod_{r=1}^{\ell}\mathscr{I}_r^{n_r}\Big\}\times(\mathbb{R}^+)^2$
- Edge function

$$\mathfrak{z}_\upsilon(\boldsymbol{p};\mathfrak{x}) = \mathcal{E}_0 + \mathfrak{x} - \sum_{r=1}^{\ell}\sum_{a=1}^{n_r}\mathfrak{u}_r(p_a^{(r)}) + \upsilon\mathrm{v}\Big\{\mathcal{P}_0 - \sum_{r=1}^{\ell}\sum_{a=1}^{n_r}p_a^{(r)}\Big\}$$

- $|\mathfrak{u}_r'|>0$ on $\mathscr{I}_r$ $\qquad \pm\mathrm{v}\notin\mathfrak{u}_r'\big(\mathrm{Int}(\mathscr{I}_r)\big)$

$$\begin{cases} \mathscr{I}_r = \mathscr{I}_r^{(\mathrm{in})}\sqcup\mathscr{I}_r^{(\mathrm{out})} \\ t_r : \mathscr{I}_1 \to \mathscr{I}_r^{(\mathrm{in})}\end{cases} \quad \text{with} \quad \begin{cases}\mathfrak{u}_r'\big(\mathrm{Int}(\mathscr{I}_r^{(\mathrm{out})})\big)\cap\mathfrak{u}_1'\big(\mathrm{Int}(\mathscr{I}_1)\big)=\emptyset \\ \mathfrak{u}_1'(k) = \mathfrak{u}_r'\big(t_r(k)\big)\end{cases}$$

- Macroscopic momentum and energy on $\mathscr{I}_1$,

$$\mathcal{P}(k) = \sum_{r=1}^{\ell} n_r\, t_r(k) \quad \text{and} \quad \mathcal{E}(k) = \sum_{r=1}^{\ell} n_r\mathfrak{u}_r\big(t_r(k)\big) \qquad \mathcal{P}'(k)\neq 0 \quad \text{on} \quad \mathrm{Int}(\mathscr{I}_1)$$

Asymptotic behaviour model integral

Theorem '18 K.

a) The regular case.

⊛ If $\big(\mathcal{P}_0,\mathcal{E}_0\big) \notin \Big\{\big(\mathcal{P}(k),\mathcal{E}(k)\big) \;:\; k\in\mathscr{I}_1\Big\}$ and $\min_{\substack{b\in\partial\mathscr{I}_1\\ \upsilon=\pm}} \big|\mathcal{E}_0 - \mathcal{E}(b) + \upsilon \mathrm{v}(\mathcal{P}_0 - \mathcal{P}(b))\big| > 0$

$\mathfrak{x}\mapsto \mathcal{I}(\mathfrak{x})$ is smooth at $\mathfrak{x}=0$

b) The singular case.

⊛ If $\big(\mathcal{P}_0,\mathcal{E}_0\big) = \big(\mathcal{P}(k_0),\mathcal{E}(k_0)\big)\,,\; k_0\in \mathrm{Int}(\mathscr{I}_1)$ and $\vartheta\notin\mathbb{N}$,

$$\mathcal{I}(\mathfrak{x}) = C\cdot|\mathfrak{x}|^{\vartheta}\cdot\bigg\{\Xi(\mathfrak{x})\frac{\sin[\pi\nu_+]}{\pi} + \Xi(-\mathfrak{x})\frac{\sin[\pi\nu_-]}{\pi}\bigg\} + \mathfrak{r}(\mathfrak{x}) + \mathrm{O}\big(|\mathfrak{x}|^{\vartheta+\alpha}\big)$$

$$C = \mathcal{G}\big(\boldsymbol{t}(k_0)\big)\cdot\frac{\Gamma(\delta_+)\Gamma(\delta_-)\Gamma(-\vartheta)\cdot(2\mathrm{v})^{\delta_++\delta_--1}}{\sqrt{|\mathcal{P}'(k_0)|}\cdot\prod\limits_{\upsilon=\pm}\big|\mathrm{v}-\upsilon\mathfrak{u}_1'(k_0)\big|^{\delta_\upsilon}}\cdot\prod_{r=1}^{\ell}\Bigg\{\frac{G(2+n_r)\cdot(2\pi)^{\frac{n_r-\delta_{r,1}}{2}}}{\big|\mathfrak{u}_r''(t_r(k_0))\big|^{\frac{1}{2}(n_r^2-\delta_{r,1})}}\Bigg\} \quad \text{with} \quad \delta_\upsilon = \Delta_\upsilon(\boldsymbol{t}(k_0))$$

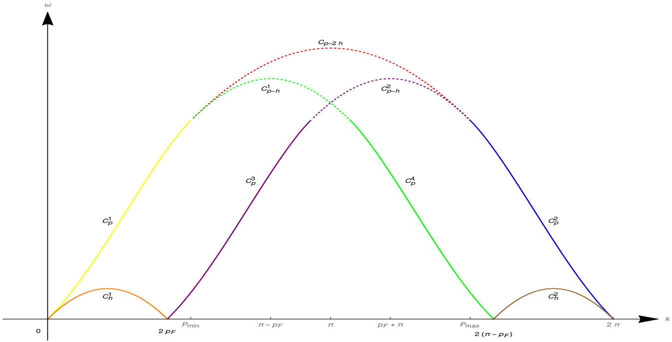
$$\vartheta = \frac{1}{2}\sum_{r=1}^{\ell} n_r^2 - \frac{3}{2} + \delta_+ + \delta_- \,, \qquad \nu_\pm = \frac{1}{2}\sum_{\substack{r=1:\\ \varepsilon_r=\mp1}}^{\ell} n_r^2 - \frac{1\mp\mathfrak{s}}{4} + \sum_{\substack{\upsilon=\pm:\\ \pm[\mathrm{v}-\upsilon\mathfrak{u}_1'(k_0)]>0}}\delta_\upsilon \quad \text{and} \quad \begin{cases} \mathfrak{s} = -\mathrm{sgn}\Big[\frac{\mathcal{P}'(k_0)}{\mathfrak{u}_1''(k_0)}\Big] \\ \varepsilon_r = -\mathrm{sgn}\big[\mathfrak{u}_r''(t_r(k_0))\big] \end{cases}$$

$$\boldsymbol{t}(k_0) = \big(\boldsymbol{t}_1(k_0),\dots,\boldsymbol{t}_\ell(k_0)\big)\in\mathbb{R}^{\overline{\boldsymbol{n}}_\ell} \quad \text{with} \quad \boldsymbol{t}_r(k_0) = \big(t_r(k_0),\dots,t_r(k_0)\big)\in\mathbb{R}^{n_r}$$

The two-particle-hole excitation thresholds

- Form factor series representation for DRF

$$\mathscr{S}^{(\gamma)}(k,\omega) = \sum_{\boldsymbol{n}\in\mathfrak{S}} \mathscr{S}^{(\gamma)}_{\boldsymbol{n}}(k,\omega)$$



The vicinity of a hole threshold

(k,ω) configuration close to the hole excitation line

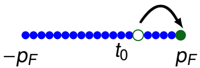
$$(\mathcal{P}_0,\mathcal{E}_0) = (p_F - t_0\,, -\mathfrak{e}_1(t_0)) \qquad \text{with} \qquad t_0 \in \,]-p_F\,;p_F\,[\ .$$

- The *hole* threshold

$$\mathcal{S}_h^{(z)}\big(\mathcal{P}_0,\mathcal{E}_0+\delta\omega\big) = \frac{(2\pi)^2\cdot\Xi(\delta\omega)\cdot[\delta\omega]^{\Delta^{(h)}}}{[\mathfrak{v}(t_0)+\mathrm{v}_F]^{\delta_+^{(h)}}\,[\mathrm{v}_F-\mathfrak{v}(t_0)]^{\delta_-^{(h)}}}\cdot\frac{\big|\mathcal{F}_h^{(z)}(t_0)\big|^2}{\Gamma(\delta_+^{(h)}+\delta_-^{(h)})}\Big(1+\mathrm{O}\big([\delta\omega]^{1-0^+}\big)\Big)+\mathcal{S}_{h;\mathrm{reg}}^{(zz)}(\delta\omega)\,.$$

- $\mathfrak{v}(t_0)=\mathfrak{e}_1'(t_0)$: velocity of the hole at t_0 $\qquad$ v_F: velocity excitations on Fremi boundary.
- Edge exponent: $\Delta^{(h)} = \delta_+^{(h)} + \delta_-^{(h)} - 1$
- $\delta_\pm^{(h)}$: microscopic shift of right(+)/left(-) Fermi boundary due to excitation.

$$\big|\mathcal{F}_h^{(z)}(t_0)\big|^2 = \lim_{L\to+\infty}\left\{\Big(\frac{L}{2\pi}\Big)^{\delta_+^{(h)}+\delta_-^{(h)}+1}\frac{\big|\big(\Omega,\sigma_1^z\Upsilon_{t_0}\big)\big|^2}{\|\Omega\|^2\cdot\|\Upsilon_{t_0}\|^2}\right\}$$



- ⋆ ground state Ω
- ⋆ excitation Υ_{t_0} $\begin{cases}\mathcal{E}_0 = -\mathfrak{e}_1(t_0)\\ \mathcal{P}_0 = p_F - t_0\end{cases}$

The vicinity of a particle threshold

(k,ω) configuration close to the particle excitation line

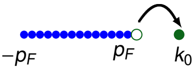
$$(\mathcal{P}_0,\mathcal{E}_0) = (k_0 - p_F\,,\mathfrak{e}_1(k_0)) \qquad \text{with} \qquad k_0 \in \,]\,p_F\,;K_m\,[\;.$$

- $\mathrm{v}_F < \mathfrak{v}(k)$ on $]\,p_F\,;K_m\,[$ *and* $\mathfrak{v}(K_m) = \mathrm{v}_F$
- The *particle* threshold

$$\mathscr{S}_p^{(zz)}\Big(\mathcal{P}_0,\mathcal{E}_0+\delta\omega\Big) = \frac{(2\pi)^2\,[\delta\omega]^{\Delta^{(p)}}\,\Gamma(-\Delta^{(p)})\cdot|\mathcal{F}_p^{(z)}(k_0)|^2}{[\mathfrak{v}(k_0)+\mathrm{v}_F]^{\delta_+^{(p)}}\,[\mathrm{v}_F-\mathfrak{v}(k_0)]^{\delta_-^{(p)}}} \times \Big\{\Xi(\delta\omega)\,\frac{\sin[\pi\delta_+^{(p)}]}{\pi} + \Xi(-\delta\omega)\,\frac{\sin[\pi\delta_-^{(p)}]}{\pi}\Big\}\Big(1+\mathrm{O}\big([\delta\omega]^{1-0^+}\big)\Big) + \mathscr{S}_{p;\mathrm{reg}}^{(zz)}(\delta\omega)\,.$$

- $\mathfrak{v}(k_0) = \mathfrak{e}_1'(k_0)$: velocity of the particle at k_0
- Edge exponent: $\Delta^{(p)} = \delta_+^{(p)} + \delta_-^{(p)} - 1$

$$\Big|\mathcal{F}_h^{(z)}(t_0)\Big|^2 = \lim_{L\to+\infty}\left\{\Big(\frac{L}{2\pi}\Big)^{\delta_+^{(p)}+\delta_-^{(p)}+1}\frac{\Big|\Big(\Omega,\,\sigma_1^z\Upsilon_{k_0}\Big)\Big|^2}{\big|\big|\Omega\big|\big|^2\cdot\big|\big|\Upsilon_{k_0}\big|\big|^2}\right\}$$



★ ground state Ω

★ excitation Υ_{k_0} $\begin{cases} \mathcal{E}_0 = \mathfrak{e}_1(k_0) \\ \mathcal{P}_0 = k_0 - p_F \end{cases}$

Multi particle-hole thresholds

- Particle and holes may have equal velocities $\mathfrak{v}(k)=\mathfrak{v}(t(k))$ $t:]\,K_m\,;K_M\,[\;\rightarrow\;]-p_F\,;p_F\,[$
  (k,ω) configuration close to an *equal*-velocity particle-hole excitations

$$(\mathcal{P}_0,\mathcal{E}_0)=\Big(n_p k_0-n_h t(k_0)+(n_h-n_p)p_F\,,\,n_p\mathfrak{e}_1(k_0)-n_h\mathfrak{e}_1(t(k_0))\Big)\qquad\text{with}\qquad k_0\,\in\,]\,K_m\,;K_M\,[\;.$$

- The *particle* threshold

$$\mathscr{S}^{(zz)}_{ph}\big(\mathcal{P}_0,\mathcal{E}_0+\delta\omega\big)=\frac{\big|\mathcal{F}^{(z)}_p(k_0)\big|^2}{\sqrt{n_p-n_h t'(k_0)}}\cdot\frac{(2\pi)^{\frac{n_p+n_h-1}{2}}G(1+n_p)G(1+n_h)}{\Big(\sqrt{-\mathfrak{v}'(k_0)}\Big)^{n_p^2-1}\Big(\sqrt{\mathfrak{v}'(t(k_0))}\Big)^{n_h^2}}\,\frac{[\delta\omega]^{\Delta^{(ph)}}\,\Gamma(-\Delta^{(p)})}{[\mathfrak{v}(k_0)+\mathrm{v}_F]^{\delta_+^{(ph)}}\,[\mathrm{v}_F-\mathfrak{v}(k_0)]^{\delta_-^{(ph)}}}$$

$$\times\Big\{\frac{\Xi(\delta\omega)}{\pi}\sin\Big[\pi\Big(\delta_+^{(ph)}+\delta_-^{(ph)}\Big)\Big]+\frac{\Xi(-\delta\omega)}{\pi}\sin\Big[\tfrac{\pi}{2}\big(n_p^2+n_h^2-3\big)\Big]\Big\}\cdot\Big(1+\mathrm{O}\Big([\delta\omega]^{1-0^+}\Big)\Big)+\mathscr{S}^{(zz)}_{ph;\mathrm{reg}}(\delta\omega)$$

- Edge exponent: $\Delta^{(ph)}=\delta_+^{(ph)}+\delta_-^{(ph)}+\frac{1}{2}\Big(n_p^2+n_h^2-3\Big)$

Conclusion and perspectives

Review of the results

- ✔ Rigourous analysis of auxiliary integrals;
- ✔ Confirms predictions of non-linear Luttinger-liquid structure (hole, particle, strings);
- ✔ Exhibits role of equal velocity excitations;
- ✔ Phenomenological form of correlators for the Luttinger liquid universality class;
- ✔ universality $\equiv$ singularity structure of form factors & *classical* saddle-point calculation;

Further developments

- ⊛ Long-time & large-distance asymptotics of dynamical two-point functions.
- ⊛ Models for $c \neq 1$.
- ⊛ Develop a saddle-point/singularity based classification of universality classes.