Exact effective interactions in string vacua with extended SUSY

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Exact effective interactions in string theory I

- Scattering amplitudes in string theory are in principle computable at weak coupling via the genus expansion. The resulting series $\sum_{h\geq 0} \mathcal{A}_h\, g_s^{2h-2}$ is asymptotic and misses non-perturbative effects of order e^{-1/g_s} associated to D-instantons. *[Shenker 1990]*
- At low energy around SUSY vacua, the dynamics of massless modes is effectively described by supergravity, corrected by an infinite series of higher-derivative effective interactions, weighted by increasing powers of $\alpha' \sim 1/l_P^2$.
- The coefficient of each effective interaction is a function $\mathcal{E}(\varphi)$ (or more generally a tensor) on the moduli space $\mathcal{M}_D$, which specifies the internal manifold $X_{d=10-D}$ as well as the string coupling g_s.
- Different cusps of $\mathcal{M}_D$ correspond to different degenerations of X_d, or to possibly different perturbative expansions related by string dualities.

Exact effective interactions in string theory II

- In string vacua with extended supersymmetry, the moduli space $\mathcal{M}_D$ is locally a symmetric space G_D/K_D, and the coefficients $\mathcal{E}(\varphi)$ are believed to be invariant (or covariant) under the action $\varphi \to g \cdot \varphi$ of an arithmetic subgroup $G_D(\mathbb{Z}) \subset G_D$.
- In toroidal compactifications, $G_D(\mathbb{Z})$ includes the T-duality $O(d, d, \mathbb{Z})$, but may also contain S-duality or U-duality generators inverting g_s or mixing it with geometric moduli.
- The coefficients $\mathcal{E}(\varphi)$ must then be automorphic forms on $\mathcal{M}_D = G_D(\mathbb{Z})\backslash G_D/K_D$, which are extensively studied by mathematicians.

Exact effective interactions in string theory III

- Supersymmetry further requires that the lowest terms in the α' expansion satisfy closed systems of differential equations on $\mathcal{M}_D$. These SUSY Ward identities often allow perturbative corrections at only few low orders, and restrict the form of non-perturbative contributions.
- Typically, effective interactions with k derivatives (or $2k$ fermions) can only be corrected by instantons carrying $2k$ fermionic zero-modes, i.e. breaking a fraction $2k/N_Q$ of the supercharges preserved the vacuum. Such terms are known as BPS couplings.
- Combining information from perturbative computations, SUSY Ward identities and duality, it is often possible to determine the coefficient $\mathcal{E}(\varphi)$ of BPS couplings exactly throughout $\mathcal{M}_D$.

Exact effective interactions in string theory IV

- Such exact results provide invaluable window into the non–perturbative regime of string theory, allowing precision tests of string dualities.
- One important application is to precision counting of BPS black holes in dimension D, via their contributions to BPS couplings in dimension $D - 1$ after reduction on a circle.
- These exact results can also suggest deep new mathematical facts about automorphic forms, or about enumerative geometry of the internal space (or both).

Outline

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Four-graviton interactions in maximal SUSY I

- Over the last 20 years or so, a lot of work has gone into implementing this program in string vacua with maximal SUSY coming from type II strings compactified on a torus T^d (or M-theory compactified on T^{d+1}).

Green Gutperle Russo Vanhove Miller BP Kiritsis Obers . . .

- The leading 4-graviton effective interactions were shown to be given by Langlands-Eisenstein series of the U-duality group:

$$\mathcal{E}_{\mathcal{R}^4} = 2\zeta(3)\,\mathcal{E}^{E_{d+1}(\mathbb{Z})}_{\frac{3}{2}\lambda}\ , \quad \mathcal{E}_{D^4\mathcal{R}^4} = \zeta(5)\,\mathcal{E}^{E_{d+1}(\mathbb{Z})}_{\frac{5}{2}\lambda}$$

where λ is the highest weight of the string multiplet (133 for E_7).

Four-graviton interactions in maximal SUSY II

- Both are eigenmodes of the Laplacian on $\mathcal{M}_D$,

$$\left[\Delta - \frac{3(d+1)(d-2)}{d-8}\right] \mathcal{E}_{\mathcal{R}^4} = 0 \,, \quad \left[\Delta - \frac{5(d+2)(d-3)}{d-8}\right] \mathcal{E}_{D^4\mathcal{R}^4} = 0 \,,$$

  and in fact satisfy much stronger tensorial Ward identities which uniquely identifies them as the automorphic forms associated to the minimal and next-to-minimal representations.

Green Russo Vanhove; BP; Bossard Verschinin

- It follows that $\mathcal{E}_{\mathcal{R}^4}$ can only receive 0+1-loop + 1/2-BPS instanton corrections, while $\mathcal{E}_{D^4\mathcal{R}^4}$ can only receive 0+1+2-loop +1/4-BPS instanton corrections.

Four-graviton interactions in maximal SUSY III

- The coefficient of the next effective interaction $\mathcal{E}_{D^6\mathcal{R}^4}$ is NOT an Eisenstein series, since it must satisfy the Poisson-type equation

$$\left[\Delta - \frac{6(d+4)(d-4)}{d-8}\right] \mathcal{E}_{D^6\mathcal{R}^4} = -\left[\mathcal{E}_{\mathcal{R}^4}\right]^2$$

  This implies that $\mathcal{E}_{D^6\mathcal{R}^4}$ can only receive 0+1+2+3-loop corrections, plus 1/8-BPS instantons plus 1/2-BPS instanton-anti-instanton pairs.

  Green Russo Vanhove Miller; Bossard Verschinin

- The exact $\mathcal{E}_{D^6\mathcal{R}^4}$ was proposed in $D = 9, 10$ from a two-loop amplitude in 11D SUGRA *[Green Vanhove Russo 2005]*, in $D = 5$ by covarianzing the genus-two string amplitude *[BP2015]*, and in any $D \geq 3$ from a two-loop amplitude in exceptional SUGRA, *[Bossard Kleinschmidt 2015]* but the full expansion at the cusps remain to be worked out *[Bossard Kleinschmidt BP, in progress]*.

Outline

Four-dimensional string vacua with 16 supercharges I

- We now turn to string vacua with half-maximal supersymmetry, obtained by compactifying the heterotic or type I string on a torus, or type II strings on $K3$ times a torus. For brevity we focus on the 'maximal rank model', although our results extend to CHL models.
- The moduli space in $D = 4$ is given by

$$\mathcal{M}_4 = \frac{SL(2)}{U(1)} \times \frac{O(22,6)}{O(22) \times O(6)}$$

  where the first factor is the heterotic axiodilaton $S = a + \mathrm{i}/g_4^2$, and the second are the heterotic Narain moduli.
- These 4D models are believed to be invariant under $G_4(\mathbb{Z})$, an arithmetic subgroup of $SL(2) \times O(22,6)$ preserving the charge lattice $\Lambda_{em} = \Lambda_e \oplus \Lambda_m$. *[Font Ibanez Lüst Quevedo 1990; Sen 1994]*

Exact BPS couplings in $D = 3$ I

- After compactification on a circle, the moduli space extends to

$$\mathcal{M}_3 = \frac{O(24,8)}{O(24)\times O(8)} \supset \begin{cases} \mathbb{R}^+_R \times \mathcal{M}_4 \times \mathbb{R}^{56+1} \\ \mathbb{R}^+_{1/g_3^2} \times \frac{O(23,7)}{O(23)\times O(7)} \times \mathbb{R}^{23+7} \end{cases}$$

Markus Schwarz 1983

- Accordingly, the U-duality group enhances to an arithmetic subgroup $G_3(\mathbb{Z}) \subset O(24,8)$, which is the automorphism group of the 'non-perturbative Narain lattice' $\Lambda_{24,8} = \Lambda_{23,7} \oplus \Lambda_{1,1}$.

Sen 1994

- We focus on the 4-derivative and 6-derivative couplings in $D = 3$

$$F_{abcd}(\Phi)\,\nabla\Phi^a\nabla\Phi^b\nabla\Phi^c\nabla\Phi^d + G_{ab,cd}(\Phi)\nabla(\nabla\Phi^a\nabla\Phi^b)\,\nabla(\nabla\Phi^c\nabla\Phi^d)$$

Exact BPS couplings in $D = 3$ II

- SUSY requires that the tensorial coefficients $F_{abcd}(\Phi)$ and $G_{ab,cd}$ satisfy various differential constraints. Among them, schematically,

$$\mathcal{D}^2_{ef}\, F_{abcd} = 0\ , \quad \mathcal{D}^2_{ef}\, G_{ab,cd} = F_{abk(e}\, F_{f)cd}{}^{k}$$

  where $\mathcal{D}^2_{ef}$ is a second order differential operator on $\mathcal{M}_3$.
- These constraints imply that F_{abcd} receives only 0+1-loop +1/2-BPS instanton corrections in heterotic perturbation theory, while $G_{ab,cd}$ receives only 0+1+2-loop+1/4-BPS instanton corrections, plus pairs of 1/2-BPS instanton-anti-instantons.

Bossard, Cosnier-Horeau, BP, 2016

Exact $(\nabla\Phi)^4$ coupling in $D=3$ I

- The coupling $(\nabla\Phi)^4$ is a 3D version of the F^4 coupling analyzed long ago. Up to non-perturbative effects,

$$g_3^2\, F_{abcd} = \frac{c_0}{g_3^2}\,\delta_{(ab}\delta_{cd)} + \mathrm{RN}\int_{\mathcal{F}_1}\frac{\mathrm{d}\rho_1\mathrm{d}\rho_2}{\rho_2^2}\,\frac{\Gamma_{\Lambda_{23,7}}[P_{abcd}]}{\Delta(\rho)} + \mathcal{O}(e^{-1/g_3^2})$$

  where $\Gamma_{\Lambda_{23,7}}$ is the partition function of the perturbative Narain lattice with polynomial insertion,

$$\Gamma_{\Lambda_{p,q}}[P_{abcd}] = \rho_2^{q/2}\sum_{Q\Lambda} P_{abcd}(Q_L)\,e^{\mathrm{i}\pi Q_L^2\rho - \mathrm{i}\pi Q_R^2\bar\rho}$$

  Lerche Nilsson Schellekens Warner 1988

- $\mathcal{F}_1$ is the standard fundamental domain of $SL(2,\mathbb{Z})$ on $\mathcal{H}_1$, and RN indicates a specific regularization of infrared divergences.

Exact $(\nabla\Phi)^4$ coupling in $D=3$ II

- Requiring invariance under U-duality, it is natural to conjecture that the exact coefficient of the $(\nabla\Phi)^4$ in $D=3$ is *[Obers BP 2000]*

$$F_{abcd} = \mathrm{RN} \int_{\mathcal{F}_1(N)} \frac{\mathrm{d}\rho_1 \mathrm{d}\rho_2}{\rho_2^2} \frac{\Gamma_{\Lambda_{24,8}}[P_{abcd}]}{\Delta}$$

- This satisfies $\mathcal{D}_{ef}^2 F_{abcd} = 0$. The limit $g_3 \to 0$ can be extracted using the orbit method, and reproduces the tree-level and one-loop terms, plus instantons from NS5 and KK5-branes.
- In the large radius limit, one finds (schematically)

$$\begin{aligned} F_{\alpha\beta\gamma\delta} =& R^2 \left(f_{\mathcal{R}^2}(S)\, \delta_{(\alpha\beta}\delta_{\gamma\delta)} + F^{(22,6)}_{\alpha\beta\gamma\delta} \right) \\ &+ \sum'_{\substack{\widetilde{Q,P}\in\Lambda_{em} \\ Q\wedge P=0}} c(Q,P)\, P_{\alpha\beta\gamma\delta}\, e^{-2\pi R\mathcal{M}(Q,P) - 2\pi \mathrm{i}(Q\cdot a_1 + P\cdot a_2)} + \mathcal{O}(e^{-R^2}) \end{aligned}$$

where $f_{\mathcal{R}^2}(S) = -\log(S_2^{12}|\Delta(S)|^2)$ and $F^{(22,6)}_{\alpha\beta\gamma\delta}$ is a similar modular integral with $\Lambda_{24,8}$ replaced by $\Lambda_{22,6}$.

Exact $(\nabla\Phi)^4$ coupling in $D = 3$ III

- The power-like terms (from the trivial orbit and zero-mode of the rank one orbit) reproduce the exact $\mathcal{R}^2$ and F^4 couplings in $D = 4$.

  Harvey Moore, Kiritsis Obers BP, 2000

- The $\mathcal{O}(e^{-R})$ terms (from the rank one orbit) correspond to 1/2-BPS dyons, weighted by $c(Q, P) = \sum_{d|(Q,P)} c\left(\frac{\gcd(Q^2,P^2,Q\cdot P)}{d^2}\right)$ where $c(N)$ are the Fourier coefficients of $1/\Delta$. This agrees with the BPS index if (Q, P) is primitive.
- The $\mathcal{O}(e^{-R^2})$ terms (from the rank two orbit) have the expected form of Taub-NUT instantons.

Exact $\nabla^2(\nabla\Phi)^4$ coupling in $D=3$ I

- The $\nabla^2(\nabla\Phi)^4$ coupling is a 3D version of the D^2F^4 coupling. Perturbatively, it receives up to two-loop corrections,

$$g_3^6\, G_{\alpha\beta,\gamma\delta} = \frac{c_0'}{g_3^2}\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\beta} G^{(23,7)}_{\gamma\delta} + g_3^2\, G^{(23,7)}_{\alpha\beta,\gamma\delta} + \mathcal{O}(e^{-1/g_3^2})$$

where the one-loop correction is given by *[Sakai Tanii 1987]*

$$G^{(23,7)}_{ab} = \mathrm{RN}\int_{\mathcal{F}_1} \frac{\mathrm{d}\rho_1\mathrm{d}\rho_2}{\rho_2^2}\,\frac{\widehat{E}_2\,\Gamma_{\Lambda_{23,7}}[P_{ab}]}{\Delta}\ ,$$

while the two-loop correction is *[d'Hoker Phong 2005]*,

$$G^{(23,7)}_{ab,cd} = \mathrm{RN}\int_{\mathcal{F}_2} \frac{\mathrm{d}^3\Omega_1\mathrm{d}^3\Omega_2}{|\Omega_2|^3}\,\frac{\Gamma^{(2)}_{\Lambda_{23,7}}[R_{ab,cd}]}{\Phi_{10}}$$

Exact $\nabla^2(\nabla\Phi)^4$ coupling in $D = 3$ II

- Here, Φ_{10} is the Igusa cusp form of weight 10, $\Gamma^{(2)}_{\Lambda_{p,q}}[R_{ab,cd}]$ is the genus-two Siegel-Narain theta series

$$\Gamma^{(2)}_{\Lambda_{p,q}}[R_{ab,cd}] = |\Omega_2|^{q/2} \sum_{Q^i \in \Lambda_{p,q}^{\otimes 2}} R_{ab,cd}(Q_L)\, e^{\mathrm{i}\pi(Q^i_L \Omega_{ij} Q^j_L - Q^i_R \bar{\Omega}_{ij} Q^j_R)}$$

  and $R_{ab,cd}$ is a polynomial in Q^i_L.
- $\mathcal{F}_2$ is a fundamental domain for the action of $Sp(4,\mathbb{Z})$ on the Siegel upper-half plane $\mathcal{H}_2$.
- RN denotes a regularization procedure which removes infrared divergences, both primitive and one-loop subdivergences.

Exact $\nabla^2(\nabla\Phi)^4$ coupling in $D = 3$ III

- It is natural to conjecture that the exact coefficient of the $\nabla^2(\nabla\Phi)^4$ coupling in $D = 3$ is given by

$$\boxed{G_{ab,cd} = \mathrm{RN} \int_{\mathcal{F}_2} \frac{\mathrm{d}^3\Omega_1 \mathrm{d}^3\Omega_2}{|\Omega_2|^3} \frac{\Gamma^{(2)}_{\Lambda_{24,8}}[R_{ab,cd}]}{\Phi_{10}}}$$

- This ansatz satisfies the differential constraint $\mathcal{D}^2 G = F^2$, where the source term originates from the pole of $1/\Phi_{10}$ in the separating degeneration.
- The limit $g_3 \to 0$ can be extracted using the orbit method (extended to genus two), and reproduces the known perturbative terms, plus an infinite series of NS5/KK5-brane instantons. Similarly, the weak coupling limits in type II and type I reproduce known perturbative corrections plus D/NS5/KK5-brane instantons.

Exact $\nabla^2(\nabla\Phi)^4$ coupling in $D=3$ IV

- In the large radius limit, we find, schematically,

$$G_{\alpha\beta,\gamma\delta} = R^4\left[G^{(22,6)}_{\alpha\beta,\gamma\delta} - f_{\mathcal{R}^2}(S)\delta_{\alpha\beta}G^{(22,6)}_{\gamma\delta} + [f_{\mathcal{R}^2}(S)]^2\delta_{\alpha\beta}\delta_{\gamma\delta}\right] + G^{(1/2)}_{\alpha\beta,\gamma\delta} + G^{(1/4)}_{\alpha\beta,\gamma\delta} + G^{(TN)}_{\alpha\beta,\gamma\delta}$$

- The $\mathcal{O}(R^4)$ term (from trivial orbit and zero-mode of rank one orbits) predicts the exact $\nabla^2 F^4$ and $\mathcal{R}^2 F^2$ couplings in $D=4$.
- The terms $G^{(1/2)}$ and $G^{(1/4)}$ (from the Abelian rank 1 and 2 orbits) come from $1/2$-BPS and $1/4$-BPS black holes in $D=4$, and are both $\mathcal{O}(e^{-2\pi R\mathcal{M}(Q,P)})$.
- The term $G^{(TN)}$ (from the non-Abelian rank 2 orbit) is $\mathcal{O}(e^{-R^2})$ and can be ascribed to Taub-NUT instantons.

- In $G^{(1/4)}$, the domain $\mathcal{F}_2$ can be unfolded to $\mathcal{P}_2 \times T^3$, where $\mathcal{P}_2$ is the space of positive definite matrices Ω_2. The integral over Ω_1 extracts the Fourier coefficient of $1/\Phi_{10}$,

$$C\left[\begin{pmatrix} -\frac{1}{2}|Q_1|^2 & -Q_1\cdot Q_2 \\ -Q_1\cdot Q_2 & -\frac{1}{2}|Q_2|^2 \end{pmatrix}; \Omega_2\right] = \int_{[0,1]^3} \mathrm{d}\rho_1 \mathrm{d}\sigma_1 \mathrm{d}v_1 \frac{e^{\mathrm{i}\pi(\rho Q_1^2 + \sigma Q_2^2 + 2vQ_1\cdot Q_2)}}{\Phi_{10}(\rho,\sigma,v)}$$

  which is a locally constant function of Ω_2.

- For large R, the integral over Ω_2 is dominated by a saddle point at

$$\Omega_2^\star = \frac{R}{\mathcal{M}(Q,P)} A^\intercal \left[\frac{1}{S_2}\begin{pmatrix} 1 & S_1 \\ S_1 & |S|^2 \end{pmatrix} + \frac{1}{|P_R\wedge Q_R|}\begin{pmatrix} |P_R|^2 & -P_R\cdot Q_R \\ -P_R\cdot Q_R & |Q_R|^2 \end{pmatrix}\right] A\,.$$

  where $\binom{Q}{P} = A\binom{Q_1}{Q_2}$, $A \in M_2(\mathbb{Z})/GL(2,\mathbb{Z})$.

Exact $\nabla^2(\nabla\Phi)^4$ coupling in $D = 3$ VI

- Approximating $C[M;\Omega_2]$ by its saddle point value, we find (schematically)

$$G^{(2)}_{\alpha\beta,\gamma\delta} = \sum_{(Q,P)\in\Lambda'_{em}} P_{\alpha\beta,\gamma\delta}\,\mu(Q,P)\,e^{-2\pi R\mathcal{M}(Q,P)-2\pi\mathrm{i}(Q\cdot a^1+P\cdot a^2)}$$

where $\mathcal{M}(Q,P)$ is the mass of a 1/4-BPS black hole, and

$$\mu(Q,P) = \sum_{\substack{A\in M_2(\mathbb{Z})/GL(2,\mathbb{Z})\\ A^{-1}\binom{Q}{P}\in\Lambda_{22,6}^{\otimes 2}}} |A|\,C\left[A^{-1}\begin{pmatrix} -\frac12|Q|^2 & -Q\cdot P\\ -Q\cdot P & -\frac12|P|^2\end{pmatrix}A^{-\intercal};\Omega_2^\star\right]$$

- In ‘primitive’ cases where only $A = 1$ contributes, $\mu(Q,P)$ agrees with the helicity supertrace $\Omega_6(Q,P;z)$, predicted by the DVV formula with the correct contour prescription. It also refines earlier proposals for counting dyons with ‘non-primitive’ charges.

Cheng Verlinde 2007; Banerjee Sen Srivastava 2008; Dabholkar Gomes Murthy 2008

Exact $\nabla^2(\nabla\Phi)^4$ coupling in $D = 3$ VII

- Corrections come from the difference between $C\,[M;\Omega_2]$ and $C\left[M;\Omega_2^*\right]$ at large Ω_2, due to wall-crossing. These corrections are of order $e^{-2\pi R(\mathcal{M}(Q_1,P_1)+\mathcal{M}(Q_2,P_2))}$, corresponding to two-instanton effects, and are exponentially suppressed away from the wall. They are required by the source term in the differential constraint and ensure the smoothness across the wall.
- In addition, there are also contributions from the region where $\det(\Omega_2) < 1$ due to deep poles at

$$m_2 - m_1\rho + n_1\sigma + n_2(\rho\sigma - v^2) + jv = 0 \quad \text{with} \quad n_2 \neq 0$$

While the integral over Ω_1 is no longer well-defined, one can estimate that these corrections are of order $e^{-2\pi kR^2}$ and resolve the ambiguities of the sum over 1/4-BPS charges.

Outline

Conclusion I

- $\nabla^2(\nabla\Phi)^4$ couplings in $D = 3$ string vacua with 16 supercharges nicely incorporate degeneracies of 1/4-BPS dyons in $D = 4$, and explain their hidden modular invariance. They give a precise implementation of Gaiotto's idea that 1/4-BPS dyons are (U-duals of) heterotic strings wrapped on genus-two Riemann surfaces.
- In $\mathcal{N} = 2$ string vacua, the relevant BPS coupling is the metric on the vector multiplet moduli space in $D = 3$, or hypermultiplet moduli space in $D = 4$. Enforcing the existence of an isometric action of $SL(2,\mathbb{Z})$ leads to (mock) modularity constraints on generalized Donaldson Thomas invariants.

Alexandrov Banerjee Manschot BP, 2016-17 and in progress