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Thermal stability in universal adiabatic computation

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Quantum Physics and Computational Complexity

- **Local Hamiltonian problem**: it's QMA-complete to decide the ground state energy of a local H up to inverse poly precision.
- Proof uses universal computation in ground state of local H,

$$|\psi_t\rangle = U_t \dots U_1 |0^n\rangle \longrightarrow |\Psi_{\rm hist}\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} |t\rangle |\psi_t\rangle$$

- Can this complexity of ground states persist at finite temperatures?
- $|\Psi_{\rm hist}\rangle$ used to show that (ideal, noiseless) adiabatic computation can be universal. Can this construction be made fault-tolerant?
- **Today**: we combine $|\psi_{\rm hist}\rangle$ with self-correcting topological quantum memories, thereby encoding universal quantum computation into a metastable Gibbs state of a k-local Hamiltonian.

Thermally Stable Universal Adiabatic Computation

- Hamiltonian enforces circuit constraints and code constraints:

$$H_{\rm final} = H_{\rm circuit} + H_{\rm code}$$

- Begin in (noisy) ground state of $H_{\rm init}$ and linearly interpolate:

$$H(s) = (1-s)H_{\rm init} + s\ H_{\rm final}$$

- **Noise model**: low temp thermal noise, intrinsic control errors
- $H_{\rm final}$ has a metastable Gibbs state, in the sense of a self correcting quantum memory with exponentially long lifetime.
- Goal is to prepare the metastable Gibbs state of $H_{\rm final}$ so that readout + classical decoding yields the result of the computation.
- $H(s)$ is k-local for some $k = \mathcal{O}(1)$, with $\mathcal{O}(1)$ interaction degree and at most $\mathrm{poly}(n)$ terms. (Proof of principle with large overheads)

Outline

- Introduction and background
  - Quantum ground state computing
  - Universal adiabatic computation
  - Local clocks: spacetime circuit Hamiltonians
  - Self-correcting memories

- Quantum computation in thermal equilibrium
  - Local circuit Hamiltonians $\Rightarrow$ transversal operations
  - Transversal operations $\Rightarrow$ local clocks
  - Coherent classical post-processing
  - Self-correction in spacetime: dressing stabilizers
  - The 4D Fault-tolerant quantum computing laboratory
  - Analysis: symmetry and the global rotation
  - Summary and Outlook

Quantum Ground State Computing

- Highly entangled states look maximally mixed with respect to local operators. How to check quantum computation with local H?

- Kitaev solved this problem by repurposing an idea from Feynman to entangle the time steps of the computation with a "clock register":

$$|\psi_T\rangle = U_T \ldots U_1 |0^n\rangle \longrightarrow |\Psi_{\mathrm{hist}}\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} |t\rangle |\psi_t\rangle$$

- These "history states" can be checked by a local Hamiltonian:

$$H_{\mathrm{circ}} = \underbrace{|0\rangle\langle 0| \otimes \left(\sum |1\rangle\langle 1|_i \right)}_{\text{input at t = 0}} + \sum_{t=0}^{T} H_{\mathrm{prop}}(t) \quad , \quad |t\rangle = |\underbrace{11\ldots1}_{t \text{ times}} 00\ldots0\rangle$$

$$H_{\mathrm{prop}}(t) = \frac{1}{2} \left(|t\rangle\langle t| \otimes I + |t-1\rangle\langle t-1| \otimes I - |t\rangle\langle t-1| \otimes U_t - |t-1\rangle\langle t| \otimes U_t^\dagger \right)$$

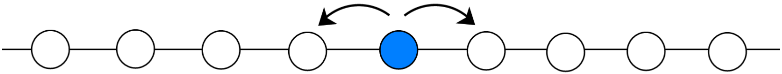
Analyzing Circuit Hamiltonians

- **Analysis:** propagation Hamiltonian is unitarily equivalent to a particle hopping on a line. Define a unitary W,

$$W = \sum_{t=0}^{T} |t\rangle\langle t| \otimes U_t ... U_1$$

- W transforms H_{prop} into a sum of hopping terms,

$$W^\dagger H_{\text{prop}} W = \sum_{t=0}^{T} \frac{1}{2} \left(|t\rangle\langle t| + |t-1\rangle\langle t-1| - |t\rangle\langle t-1| - |t-1\rangle\langle t| \right)$$



- **Diffusive random walk:** mixing time $\sim T^2$, spectral gap $\sim T^{-2}$.

Universal Adiabatic Computation

- Begin in an easily prepared ground state and slowly change H while remaining in the ground state by the adiabatic principle,

$$H(s) = (1-s)H_{\text{init}} + s\ H_{\text{final}} \quad , \quad 0 \leq s \leq 1$$

- Run-time estimate: $\sim \|\dot{H}\|/\Delta_{\min}^{-2}$, where $\Delta = \min_s gap(H(s))$.

- Universal AQC: $H_{\text{final}} = H_{\text{init}} + H_{\text{prop}}$

- Monotonicity argument shows that the minimum spectral gap occurs at $s = 1$, so $\Delta \approx T^{-2}$ and overall run time is polynomial in n, T.

- Perturbative gadgets enable universal AQC with 2-local H,

$$H = \sum_i h_i Z_i + \sum_i \Delta_i X_i + \sum_{i,j} J_{i,j} Z_i Z_j + \sum_{i,j} K_{i,j} X_i X_j$$

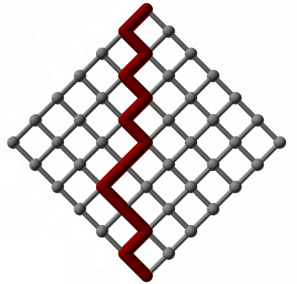
History States with Local Clocks

- Instead of propagating every qubit according to a global clock, assign local clock registers to the individual qubits,

$$|\boldsymbol{\tau}\rangle = |t_1 ... t_n\rangle \quad , \quad |\Psi_{\text{hist}}\rangle = \sum_{\boldsymbol{\tau}\ \text{valid}} |\boldsymbol{\tau}\rangle |\psi(\boldsymbol{\tau})\rangle$$

- Makes history state Hamiltonians more realistic for 2D AQC (Gosset, Terhal, Vershynina, 2014. Lloyd and Terhal, 2015).

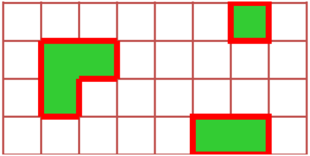
Instead of a hopping particle, the Hamiltonian is unitarily equivalent to the diffusion of a string or membrane.



Classical Self-Correcting Memories

- Ferromagnets and repetition codes: the Ising model
- **1D Ising model**: thermal fluctuations can flip a droplet of spins, energy cost is independent of the size of the droplet

- **2D Ising model**: energy cost of droplet proportional to boundary,



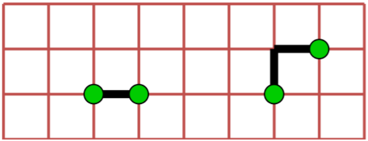
- At temperature T droplets of size L are supressed by $e^{-L/T}$. Ferromagnetic order at $T < T_c$, magnetization close to $\pm n$.
- Robust storage of classical information: lifetime scales exponentially in the size of the block. Hard disk drives work at room temperature.

Topological Quantum Error Correction

- Quantum codes require local indistinguishability $\implies$ topological order (toric code) instead of symmetry-breaking order (Ising model).

$$H_{\text{code}} = -\sum_{s \in \mathcal{S}} H_s \quad , \quad \mathcal{S} = \{ \text{ stabilizer generators } \}$$

- 2D toric code analogous to 1D Ising model: thermal fluctuations create pairs of anyons connected by a string. No additional cost to growing the string $\implies$ constant energy cost for a logical error.



- 4D toric code: logical operators are 2D membranes, energy cost scales like the 1D boundary so errors supressed by $e^{-L/T}$.
- **Open question**: finite temperature topological order in 3D?

Challenges in Adiabatic Fault-Tolerance

- Past approaches replace bare operators X, Z with logical operators X_L, Z_L. 4-qubit code suppresses 1-local thermal noise (JFS'05).
- **Challenge**: Codes with macroscopic distance have high-weight logical operators that don't correspond to local Hamiltonian terms.
- **Solution**: use circuit Hamiltonians for gate model fault-tolerance schemes with only transversal operations and local measurements.
- **Consequence 1**: circuit-model fault-tolerance requires parallelization $\implies$ spacetime construction with local clocks.
- **Consequence 2**: there can be no universal set of transversal gates $\implies$ history state must include measurement and classical feedback.

Challenges in Adiabatic Fault-Tolerance

- **Challenge**: What is the noise model?
- **Solution**: (1) weak coupling to a Markovian thermal bath, (2) Hamiltonian coupling errors , (3) probabilistic fault-paths.
- Self-correcting memories protect against thermalization, and even turn it into an advantage by using it to erase information.
- Protection from Hamiltonian coupling errors and probabilistic fault-paths relies on gate model FT and self-correcting clocks.

Self-Correcting History States

- Each logical qubit $Q_1, ..., Q_n$ in the history state is made of physical qubits $q_{i,1}, ..., q_{i,m}$. Each physical qubit $q_{i,j}$ has its own clock $t_{i,j}$.

- Just as in the classical case, both the computation and the code stabilizers are enforced by local Hamiltonian terms.

$$H = \sum_{\boldsymbol{\tau}} H_{\mathrm{prop}}(\boldsymbol{\tau}) + \sum_{\boldsymbol{\tau}} H_{\mathrm{code}}(\boldsymbol{\tau})$$

- H_{prop} needs to consist of local gates, and H_{code} needs to accomodate the propagation of the circuit without frustration.

- Apply to any FT scheme with local code checks and local operations e.g. 2D surface code with magic state injection.

- Gate teleportation uses logical measurement and classical post-processing, which will all be part of the history state.

Transversal Unitaries in a Local Hamiltonian

- Transversal operations: $U[Q_{\text{logical}}] = \bigotimes_q U[q_{\text{physical}}]$

- Advancing all clocks in a logical qubit at once would not be local $\implies$ local clocks must be advanced independently by local terms,

$$H_U[\mathbf{t}_{Q_i}, Q_i] \longrightarrow \sum_{q_i \in Q_i} H_{\text{prop}}[t_{q_i}, q_i]$$

- Need to protect the clocks from getting far out of sync $\implies$ $H_{\text{prop}}[t_{q_i}, q_i]$ checks the neighboring clocks before advancing t_i, q_i

- Challenge: advancing clocks one at a time would violate terms in H_{code}. We solve this with "dressed stabilizers."

Dressing stabilizers to avoid frustration

- We need to tell the stabilizers "what time it is" so that they can accomodate diffusive propagation without frustration,

$$|t_{s_1}, \ldots, t_{s_m}\rangle\langle t_{s_1}, \ldots, t_{s_m}| \otimes H_s(t_{s_1}, \ldots, t_{s_m})$$

- Stabilizers acting on "staggered" time configurations rotate the qubits that are lagging behind (or getting ahead),

$$|\mathbf{t}_s\rangle\langle\mathbf{t}_s| \otimes H_s(\mathbf{t}) := \left(\bigotimes_{k\in s} |t_k\rangle\langle t_k|_{t_k}\right) \bigotimes \left(\prod_{t_k} U^\dagger_{t_k,t}[q_k]\right) H_s \left(\prod_{t_k} U_{t_k,t}[q_k]\right)$$

- Spacetime view of advanced / retarded potentials in E&M
- Dressing for two qubit gates intertwines stabilizers from distinct logical qubits, but terms remain k-local.
- Suffices to limit staggering to constant window c (speed of light). Locality and number of terms grows exponentially in c.

Everything is unitary in a larger Hilbert space

- Replace projective measurement $\Pi_0 + \Pi_1 = I$ of the physical qubits with coherent unitaries onto the classical ancillas:

$$|\psi\rangle|0\rangle \longrightarrow \Pi_0|\psi\rangle|0\rangle + \Pi_1|\psi\rangle|1\rangle$$

- Each physical qubit is measured by a "classical wire". The classical wire is a logical ancilla encoded in the repetition code.
- Tip of the wire is very pointy (local, bounded degree interactions), then grows like a concatenated tree to become macroscopic.
- Classical post-processing is global and takes poly time. The rest of the computation "waits around" for this to be done.

The 4D spacetime view of active error correction

- Consider the history state of a fault-tolerant quantum computer e.g. surface code qubits connected to a classical computer.
- Instead of a code Hamiltonian, such a scheme depends on actively measuring and correcting stabilizers.
- There is no energetic protection of the qubits, but there is energetic protection from the materials in the classical computer.
- Active error correction is possible because we dump entropy from quantum computers into classical self-correcting memories.
- In our case it suffices for H_{code} to be a repetition code acting on the (coherent) classical ancilla.

Analysis of the rotated Gibbs state

- The entire Hamiltonian is unitarily equivalent to a diffusing membrane and a static code Hamiltonian, the dressing disappears:

$$W = \sum_{\boldsymbol{\tau}\,\mathrm{valid}} U(\boldsymbol{\tau})|\boldsymbol{\tau}\rangle\langle\boldsymbol{\tau}| \quad , \quad W^\dagger H W = H_{\mathrm{membrane}} \otimes I + I \otimes H_{\mathrm{code}}$$

- Put time configurations on a circle $(U_1^\dagger \ldots U_T^\dagger U_T \ldots U_1)$, symmetry makes all valid time configurations equally likely in every eigenstate.

- Initialization: classical ancillas in logical $\bar{0}$ state protected by repetition H_{code}, computational qubits in arbitrary state.

- Metastable Gibbs state of $W^\dagger H W$ is uniform over times, maximally mixed on computational qubits, and close to $\bar{0}$ on ancillas.

Analysis of the real Gibbs state

- Diagonal elements of the thermal density matrix of H in the time register basis have the form

$$|\boldsymbol{\tau}\rangle\langle\boldsymbol{\tau}| \otimes U(\boldsymbol{\tau}) \left(\rho_{\substack{\text{encoded}\\\text{qubits}}} \otimes \rho_{\substack{\text{encoded}\\\text{ancillas}}}\right) U^\dagger(\boldsymbol{\tau})$$

- FT circuit $U(\boldsymbol{\tau})$ coherently measures and corrects syndromes to initialize the quantum code and evolve the computation.
- Correct operation of $U(\boldsymbol{\tau})$ depends on dumping entropy into the thermally stable classical logical ancillas.
- Thermal stability of the ancillas is unaffected by W (e.g. Davies generators only depend on the spectral properties of H).
- Intrinsic control errors in local H terms: $\|W_{\text{actual}} - W_{\text{ideal}}\|$ is small because W is a fault-tolerant circuit.

Summary and Outlook

- Universal quantum computation in a finite temperature state of a k-local Hamiltonian with polynomial overhead.
- 4D self-correcting memory from the history state of 3D FT-QC. Relates planar FT architectures to self-correction in 3D.
- Lower bounding the gap of H_{membrane} is an open problem in mathematical physics; to obtain a tractable gap analysis we consider nonuniform distributions of time configurations.
- Benefit of applying the scheme to smaller geometrically local architectures that may not be fully thermally stable?
- Thank you for your attention!