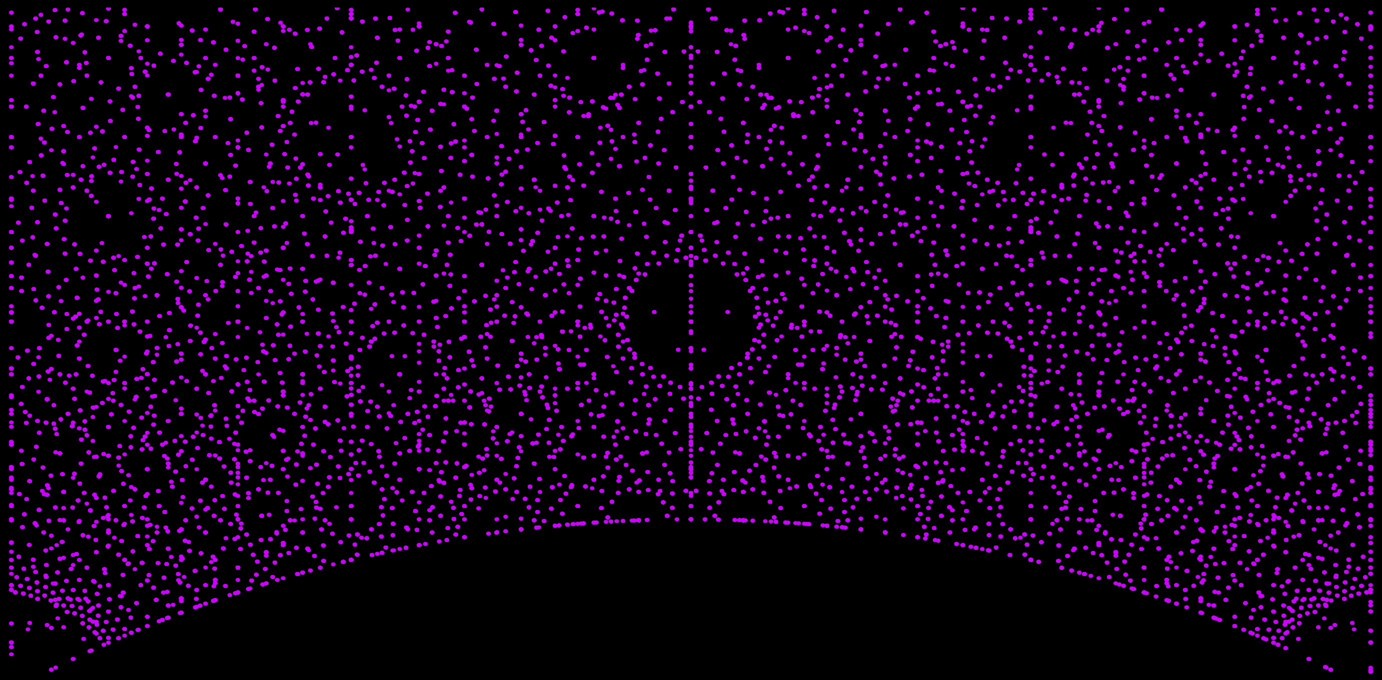
Black holes, class numbers, and special cycles



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based on work with:

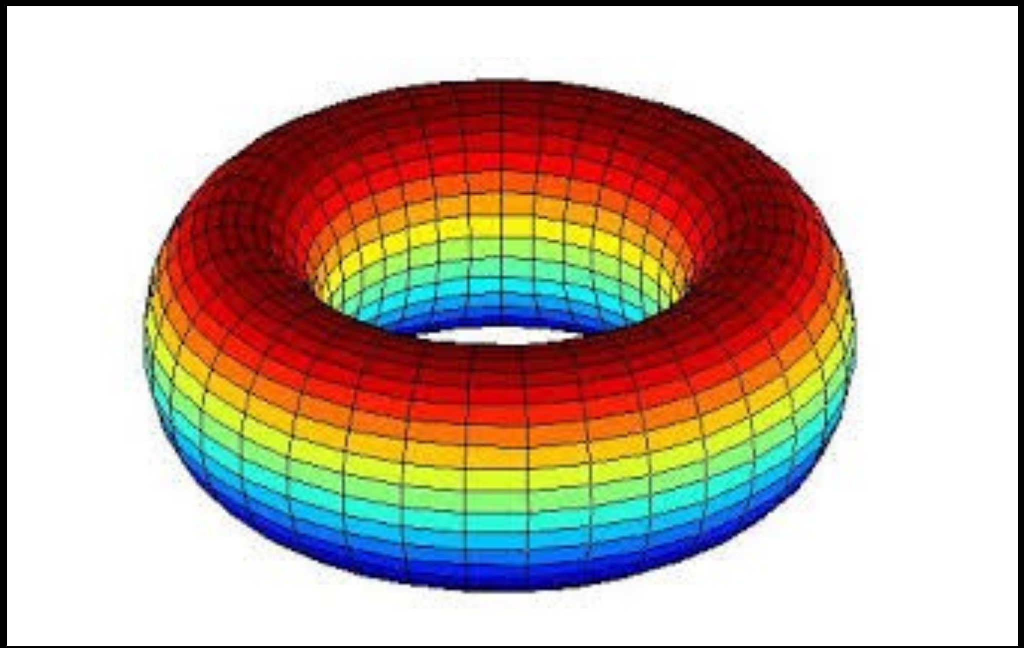
A. Tripathy (arXiv:1705.06295, 1706.02706)
N. Benjamin, K. Ono, L. Rolen (arXiv:1807.00797)

I. Introduction

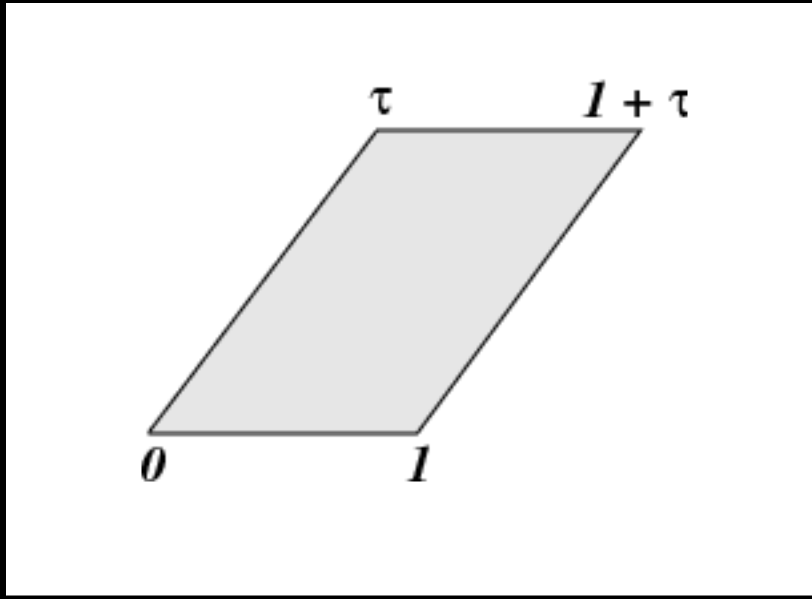
Here are two interesting questions about string theory and string compactification:

i. Some prototypical string models, such as Calabi-Yau compactifications, come with moduli spaces of vacua.

e.g. compactification on a two-torus:



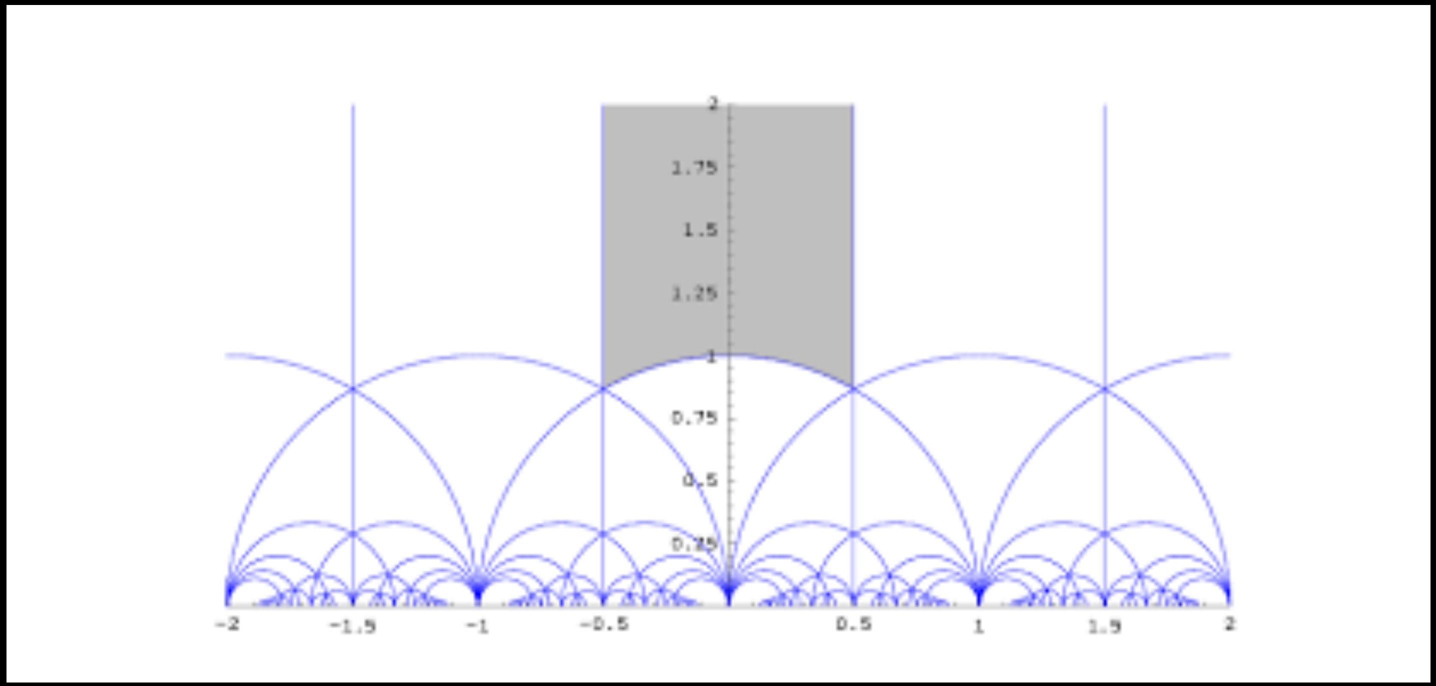
The string theory depends on the choice of complex structure on the torus.



$$\tau \simeq \frac{a\tau + b}{c\tau + d}, \quad ad - bc = 1$$

large diffeomorphisms

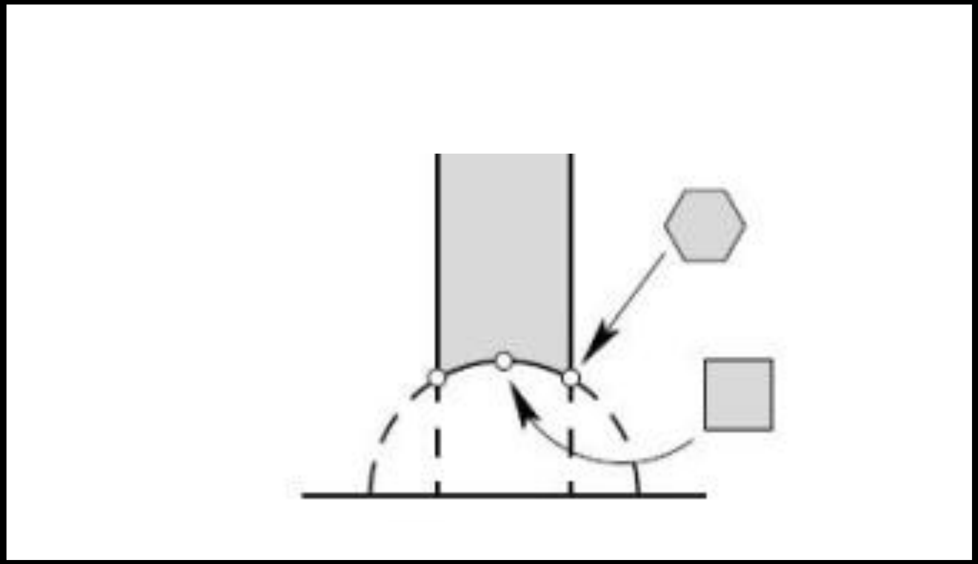
The resulting moduli space of complex structures:



Q: Are there special points in such moduli spaces?

(If so, we might think physics is interesting there, or eventually prefers them.)

In the moduli space at hand, the obvious choice would be the tori with enhanced symmetry:



This is not a bad answer, but as it only singles out two points, it isn't very rich.

There is a richer possible story!

We can consider tori with a modular parameter solving

$$a\tau^2 + b\tau + c = 0$$

$$a, b, c \in \mathbb{Z}$$

Such tori are said to admit "complex multiplication." We will see that they are physically special too, after embedding them in a more elaborate physical setting.

ii. What are the properties of black holes in string theory?

We start with the "goldilocks setting" of compactification on a Calabi-Yau threefold X.

The moduli space splits, locally, into a product:

$$\mathcal{M} = \mathcal{M}_v \times \mathcal{M}_h$$

$\mathcal{M}_v$: associated with scalars in vector multiplets

$\mathcal{M}_h$: associated with scalars in hyper multiplets

In IIB string theory on X, the vector multiplet moduli space is the moduli space of complex structures on X.

There are, in particular, abelian gauge fields in correspondence with complex structure deformations of X.

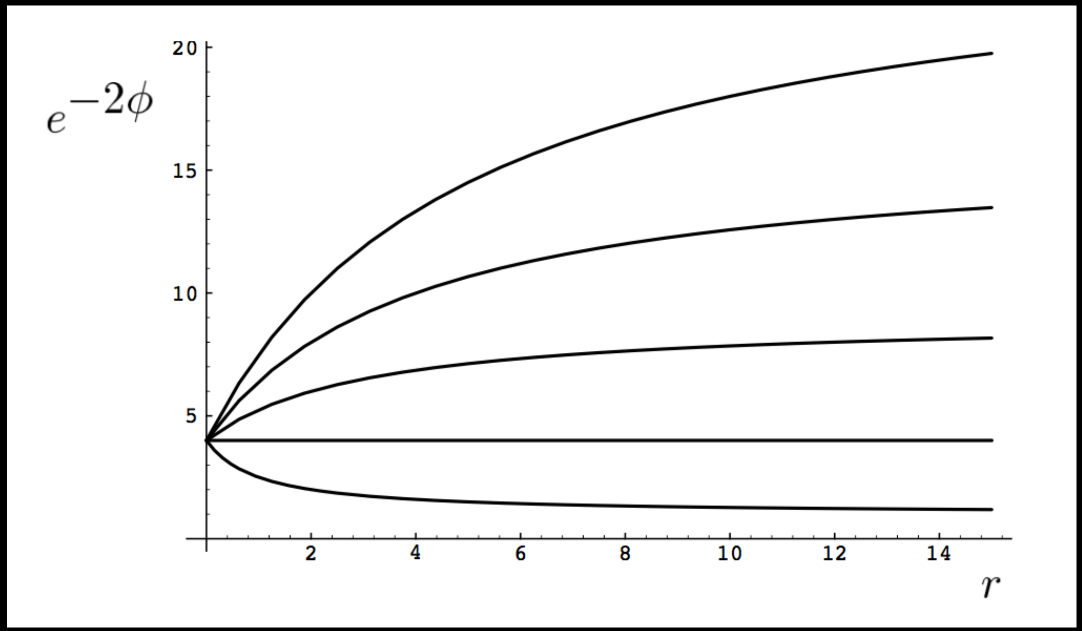
The low energy effective field theory resulting from compactification on X has a Lagrangian

$$\mathcal{L} = \int \, d^4x \; f_{ab}(\phi) F_{\mu\nu}{}^{a} F^{\mu\nu b} + \cdots$$

coupling the scalars spanning $\mathcal{M}_v$ — the complex moduli of X — to the abelian gauge fields.

Now, suppose we wish to consider a charged black hole arising in compactification on X.

Because of the coupling of the gauge fields to the scalars, we will find an effective potential for the complex moduli of X!



scalars attracted to point in moduli space that minimizes the mass.

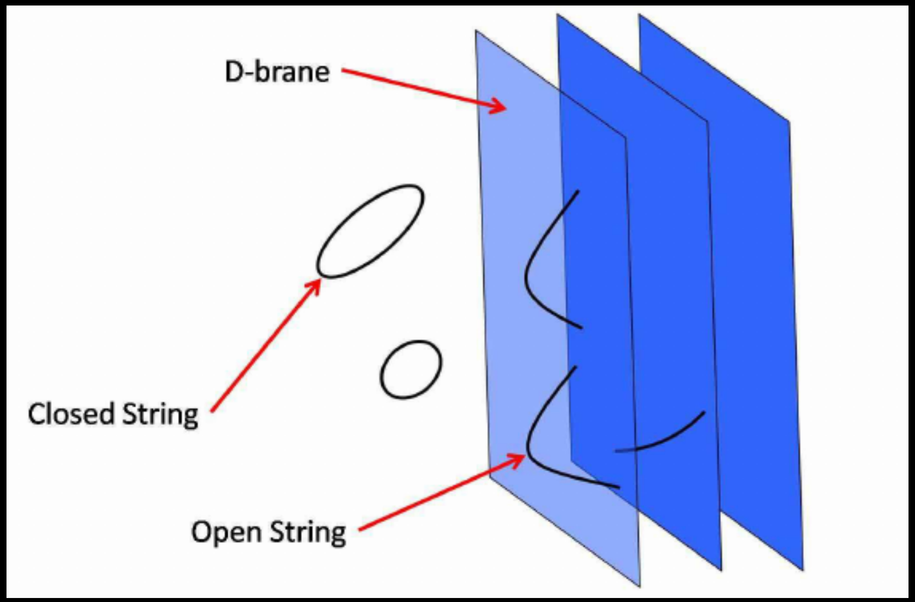
Now, we see a relation with our first question:
The attractor mechanism relates AdS2 near-horizon geometries of BPS black holes, to special points in the complex structure moduli space of X.

In the rest of the talk, we will try to learn about the nature of these special points in a simple example:

X = K3 x T2.

K3 is perhaps the simplest non-trivial compact Calabi-Yau manifold, so this is perhaps the easiest threefold to start with.

II. Attractor black holes on X = K3 x T2



We make charged black holes by wrapping D3-branes on three-cycles in the compact dimensions.

We choose a charge:

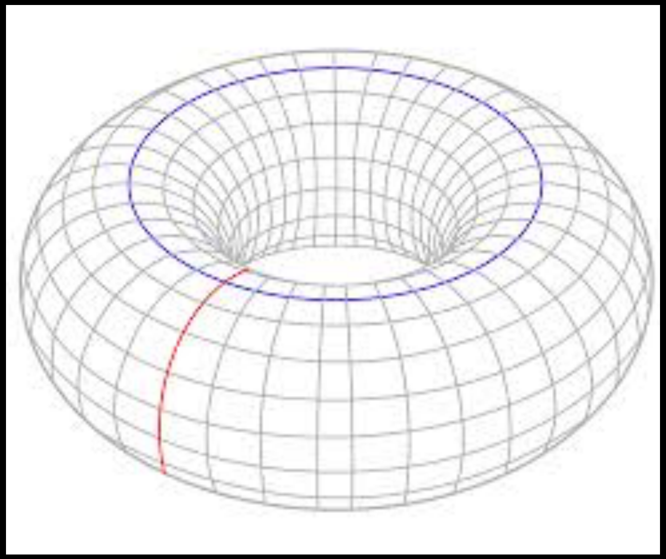
$$Q \in H^3(X, \mathbb{Z})$$

Then, we minimize:

$$Z = \int_X Q \wedge \Omega \ .$$

The result is a set of points in the complex structure moduli space of X.

For X = K3xT2, the answer is particularly elegant.



$$\alpha, \beta \in H^1(T^2)$$

$$\omega_i \in H^2(K3), \quad I = 1, \cdots, 22$$

Without loss of generality:

$$Q = \sum_i (q_i \omega_i) \wedge \alpha + (p_i \omega_i) \wedge \beta \ .$$

q, p are the electric and magnetic charge vectors

The attractor varieties have a very simple structure.

Define the Picard rank

$$\rho = \dim \left(H^{1,1}(X) \cap H^2(X, \mathbb{Z}) \right)$$

Then the attractor varieties consist of: Moore

— A "singular" K3, i.e. one with

$$\rho = 20 \ .$$

— A torus whose complex structure solves a quadratic equation with discriminant:

$$D = (p \cdot q)^2 - p^2 q^2 < 0 \ .$$

The “singular” K3 associated with a given torus is in fact determined (partly via a theorem of Shioda and Inose) by the complex structure of the torus.

To the data in our problem, we can associate a binary quadratic form:

$$\begin{pmatrix} p^2 & -p \cdot q \\ -p \cdot q & q^2 \end{pmatrix}$$

In physics, there is an SL(2,Z) duality group that acts on the electric and magnetic charges.

In mathematics, Gauss discovered a natural SL(2,Z) action on binary quadratic forms. We say two quadratic forms

$$[a, b, c] \to ax^2 + bxy + cy^2$$
$$[a', b', c'] \to a'x^2 + b'xy + c'y^2$$

are equivalent if

$$[a, b, c](x, y) = [a', b', c'](\alpha x + \beta y, \gamma x + \delta y)$$
$$\alpha\delta - \beta\gamma = 1 \ .$$

The number of SL(2,Z) inequivalent quadratic forms at a given D < 0 is called the class number associated to D. (They in fact form a group, the class group.)

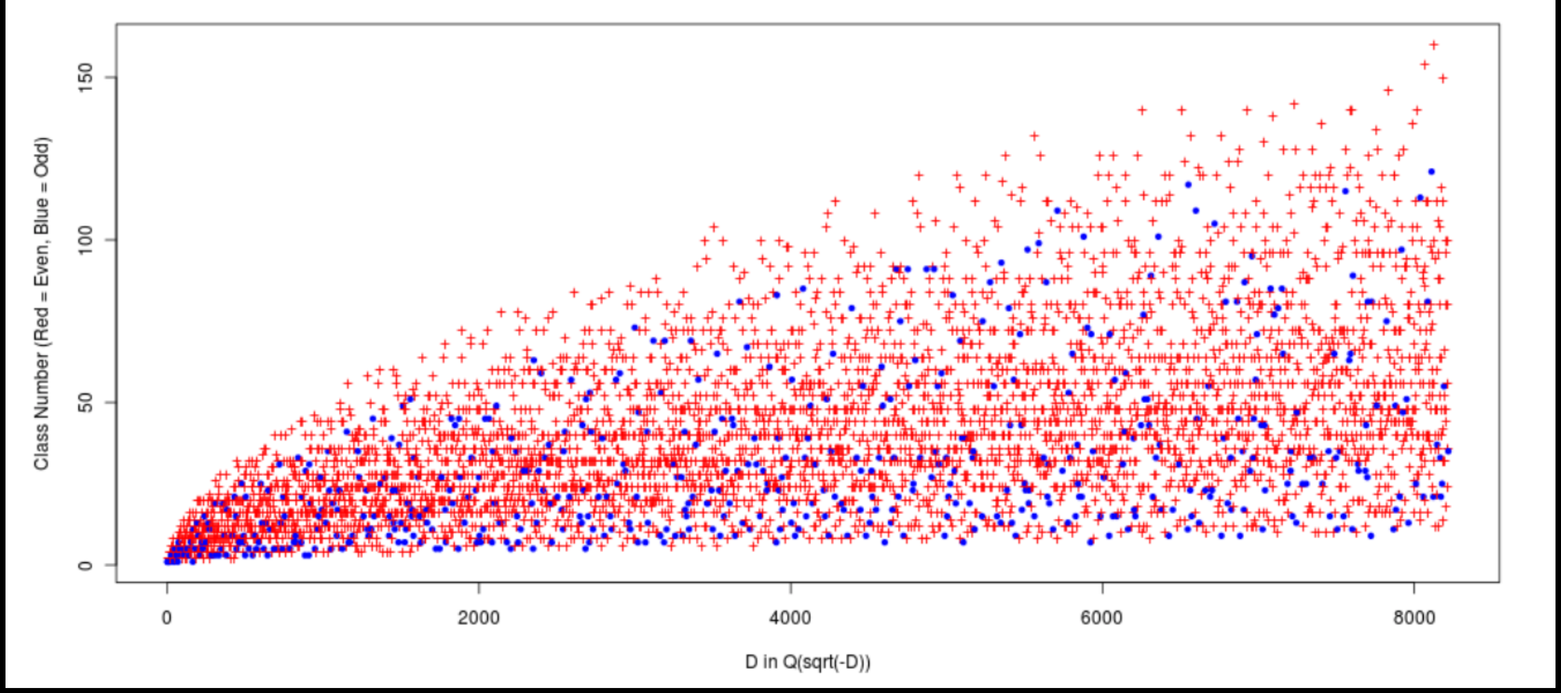
We've now seen a sketch of the logic that leads to the striking result:

Attractors on K3 x T2 ⟷ SL(2,Z) equivalence classes of binary quadratic forms

The class numbers and CM points have interesting structure!

— CM points are equidistributed.

— Determining the class numbers as a function of the discriminant has inspired great effort over the centuries. Here is a plot:



— Amazingly, the class numbers (or really the CM points) are automorphic!

Define the Hurwitz class numbers:

$H(N) = \#\{SL(2,\mathbb{Z})$ equivalence classes of possibly imprimitive quadratic forms with discriminant $-N$ weighted by the inverse order of their auto−morphism group$\}$.

Next, define the counting function:

$$Z(N) = \sum_N H(N) q^N, \quad q = e^{2\pi i \tau} \ .$$

Z is the holomorphic piece of a mock-modular form of weight 3/2!

III. Kudla-Millson theory

What we saw here is a simple example of a more general story in mathematics, that enjoys other applications to string theory.

Consider the arithmetic locally symmetric space:

$$\mathcal{M}(p,q) = O(p,q;\mathbb{Z})\backslash O(p,q;\mathbb{R})/(O(p)\times O(q)) \ .$$

Many of the most canonical moduli spaces of string compactifications take this form (often with p-q = 8k).

Heuristically, we can think of this moduli space as arising from taking a lattice of signature (p,q) and decomposing it into p left-movers and q right-movers.

Now, define “special cycles” as follows:

Choose a vector of norm -N:

$$\langle x, x \rangle = -N \ .$$

$$D_x \equiv \{\mathrm{locus\ in\ } \mathcal{M} \mathrm{\ where\ x\ is\ purely\ left-moving}\} \ .$$

These loci in the moduli space are known as “special cycles.”

What is special about them?

— They are totally geodesic submanifolds.

— In suitable string theory problems, they are the loci on moduli space where the spectrum of BPS states enhances (“jumps”).

Now, define the sum:

$$D_N \equiv \sum_{x, \langle x,x \rangle = -N} D_x \ .$$

We can consider a cohomology class on the moduli space:

$$\phi(\tau) = \sum_N \, [D_N] q^N \ .$$

For the special cases of most interest, where we consider moduli spaces associated to even unimodular lattices (p-q = 8k), there is a result:

$$\text{Kudla} - \text{Millson} : \phi(\tau) \text{is an automorphic form of weight}$$

$$\frac{p+q}{2} \text{ for } SL(2,\mathbb{Z}) \ .$$

Special cycles of higher dimension can be defined by intersections of the basic special cycles we discussed. Kudla-Millson assign Siegel forms of higher degree to these structures.

The modularity of the counting function enumerating attractors on K3xT2 (whose q-expansion has Hurwitz numbers as coefficients and black hole entropies as exponents), arises as a special case.

There are other examples where functions “counting” loci where BPS states jump are similarly automorphic.

A physics understanding of why would be nice.