Kakutani equivalence of unipotent flows

Adam Kanigowski

Montreal, 07.27.2018
(joint w. K. Vinhage and D. Wei)

General setting

- $(X, \mathcal{B}, \mu)$ – probability standard Borel space;
- $(T_t) : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ – measure-preserving, ergodic flow.

Isomorphism

$(T_t) : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$, $(S_t) : (Y, \mathcal{C}, \nu) \to (Y, \mathcal{C}, \nu)$ are isomorphic, if

$$R \circ T_t = S_t \circ R \text{ for } t \in \mathbb{R},$$

where $R : (X, \mathcal{B}, \mu) \to (Y, \mathcal{C}, \nu)$ is invertible and $\nu(C) = \mu(R^{-1}C)$, $C \in \mathcal{C}$.

Classification up to isomorphism

NOT possible in general (M. Foreman, D. Rudolph, B. Weiss, 2011)

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Orbit and Kakutani equivalence

Orbit equivalence

- (T_t) and (S_t) are orbit equivalent, if there exists a invertible transformation that maps orbits of (T_t) to orbits of (S_t).
- (Dye's theorem, 1959) Every two ergodic flows are orbit equivalent.

Kakutani equivalence, 1943

$(T_t) : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ and $(S_t) : (Y, \mathcal{C}, \nu) \to (Y, \mathcal{C}, \nu)$ are Kakutani equivalent (denoted $(T_t) \overset{K}{\sim} (S_t)$), if they have isomorphic sections, i.e. can be represented as special flows over the same transformation.

Standard flows

- Let $(R^\alpha_t) : \mathbb{T}^2 \to \mathbb{T}^2$, $R^\alpha_t(x, y) = (x + t, y + t\alpha)$. Then, for every $\alpha, \beta \notin \mathbb{Q}$, $(R^\alpha_t) \overset{K}{\sim} (R^\beta_t)$ (Katok, 1976; Ornstein, Rudolph, Weiss, 1982);
- (T_t) is standard, if $(T_t) \overset{K}{\sim} (R^\alpha_t)$ for some $\alpha \notin \mathbb{Q}$.

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Some results on Kakutani equivalence

Standard systems

- finite rank systems (Ornstein, Rudolph, Weiss, 1982);
- closed under factors, inverse limits, compact extensions (Katok, 1976; Ornstein, Rudolph, Weiss, 1982)
- horocycle flows (Ratner, 1978)

Non-standard systems

- first example due to Feldman, 1975;
- uncountably many pairwise non Kakutani equivalent systems (Ornstein, Rudolph, Weiss, 1982);
- Let (h_t) denote the horocycle flow on $SL(2,\mathbb{R})/\Gamma$. Then $(h_t)^k \overset{K}{\not\approx} (h_t)^l$ for $k \neq l$ (Ratner, 1980).

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Kakutani invariant of M. Ratner

Kakutani invariant (Ratner, 1980)

- $(T_t) \mapsto e((T_t)) \in [0, +\infty]$;
- If $(T_t) \overset{K}{\sim} (S_t)$, then $e((T_t)) = e((S_t))$.

$\bar{f}$-metric

Fix a finite partition $\mathcal{P}$ and $\epsilon > 0$. For $N > 0$, $x, y \in X$ are $(\epsilon, \mathcal{P})$-matchable for time N if there exists a set $A \subset [0, N]$, $|A| > (1-\epsilon)N$ and an increasing, measure preserving map $h : A \to h(A)$ such that $T_t x$ and $T_{h(t)} y$ are in one atom of $\mathcal{P}$ for $t \in A$.

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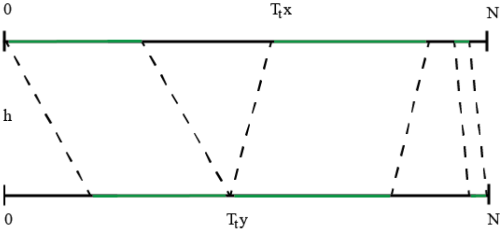

Kakutani equivalence of unipotent flows

Ratner's problem, 1994

What can be said about Kakutani equivalence for unipotent flows on quotients of semisimple Lie groups?

Setting

- G is a semisimple matrix Lie group with Lie algebra $Lie(G)$;
- $U \in Lie(G)$ is such that ad_U is nilpotent, where $ad_U : Lie(G) \to Lie(G)$, $ad_U(V) = [U, V]$;
- Γ is a uniform lattice in G;
- $\phi_t^U : G/\Gamma \to G/\Gamma$, $\phi_t^U(x\Gamma) = \exp(tU)x\Gamma$.

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Chain basis

Chain basis for unipotent elements

$U \in Lie(G)$ is a unipotent element. There exists a basis $(X_j^i)_{1 \le j \le m_i, 1 \le i \le K}$, of $Lie(G)$ such that

$$X_{m_i}^i \overset{ad_U}{\mapsto} X_{m_i-1}^i \overset{ad_U}{\mapsto} \dots \overset{ad_U}{\mapsto} X_1^i \overset{ad_U}{\mapsto} 0,$$

for every $1 \le i \le K$. In particular, $X_1^i \in C(U)$ for $1 \le i \le K$.

Growth number of U

Let

$$GR(U) := \frac{1}{2} \sum_{i=1}^{K} m_i(m_i - 1).$$

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Main theorem

Main theorem (K., Vinhage, Wei, 2018)

Let (ϕ_t^U) be a unipotent flow on G/Γ. Then

$$e((\phi_t^U)) = GR(U) - 3.$$

Moreover, if $GR(U) = 3$, then (ϕ_t^U) is standard.

Corollaries

- The only standard unipotent flows are of the form $id \times \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ acting on $(G \times SL(2, \mathbb{R}))/\Gamma$, where Γ is irreducible;
- If $dim\, G > 3$ and G is simple, then no unipotent flow on G/Γ is standard.

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Examples

Jakobson-Morozov theorem

For every unipotent $U \in Lie(G)$, there exists $V, X \in Lie(G)$ such that

$$[X, U] = 2U, \quad [X, V] = -2V, \quad [U, V] = X.$$

So, $V \mapsto X \mapsto -2U$ is always one of the chains and hence $GR(U) \geq 3$.

Examples

- $e((h_t)^k) = 3k - 3$;
- Let $U = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \in Lie(SL(3, \mathbb{R}))$. Then

$$e((\phi_t^U)) = 10.$$

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