Tau functions, Fredholm determinants and combinatorics

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Motivation: Painlevé equations

2D Ising correlation → Painlevé III (massive scaling limit) [Wu,McCoy,Tracy,Barouch,'76]

2D Ising correlation → Painlevé VI (lattice) [Jimbo,Miwa,'81]

impenetrable Bose gas → Painlevé V (sine kernel) [Jimbo,Miwa,Mori,Sato,'80]

random matrix theory → Painlevé II–VI (Airy kernel, etc) [Tracy,Widom,'92; ...]

Painlevé equations describe simplest cases of monodromy preserving deformations of linear ODEs with rational coefficients. E.g. Painlevé VI corresponds to rank 2 Fuchsian system with 4 regular singularities at $0, t, 1, \infty$:

$$\partial_z \Phi = \Phi A(z), \qquad A(z) = \frac{A_0}{z} + \frac{A_t}{z-t} + \frac{A_1}{z-1}$$

Isomonodromy equations are

$$\frac{dA_0}{dt} = \frac{[A_0, A_t]}{t}, \qquad \frac{dA_1}{dt} = \frac{[A_1, A_t]}{t-1}, \qquad A_\infty = \text{const}$$

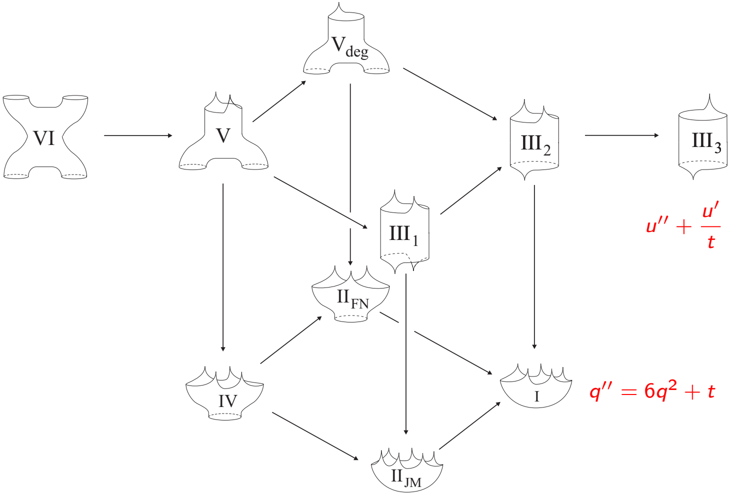
For $A_{0,t,1}$ and $A_\infty := -A_0 - A_t - A_1$ traceless 2×2 matrices, with eigenvalues $\pm\theta_{0,t,1,\infty}$, these equations are equivalent to Painlevé VI.

Painlevé VI:

$$\left(t(t-1)\zeta''\right)^2 = -2\,\mathrm{det}\begin{pmatrix} 2\theta_0^2 & t\zeta'-\zeta & \zeta'+\theta_0^2+\theta_t^2+\theta_1^2-\theta_\infty^2 \\ t\zeta'-\zeta & 2\theta_t^2 & (t-1)\zeta'-\zeta \\ \zeta'+\theta_0^2+\theta_t^2+\theta_1^2-\theta_\infty^2 & (t-1)\zeta'-\zeta & 2\theta_1^2 \end{pmatrix}$$

- $\zeta(t) = (t-1)\,\mathrm{Tr}\,A_0A_t + t\,\mathrm{Tr}\,A_1A_t = t(t-1)\frac{d}{dt}\ln\tau$
- $\tau(t)$ is the Painlevé VI tau function

Geometric confluence diagram [Chekhov, Mazzocco, Rubtsov, '15]:



$$u'' + \frac{u'}{t} = \sin u$$

$$q'' = 6q^2 + t$$

Painlevé project:

- develop a general approach that would allow to derive systematically (asymptotic) series for PI-PV functions
- explicit expressions for coefficients of the series + connection formulas (in terms of monodromy of the associated linear problem)

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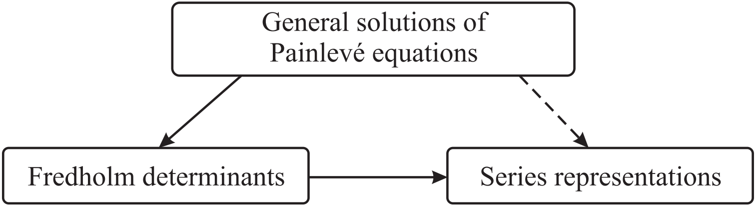
All classical "linear" special functions admit explicit representations. The Painlevé transcendents do not.

A. Fokas, A. Its, A. Kapaev, V. Novokshenov, *Painlevé transcendents. The Riemann-Hilbert approach*, (2006)

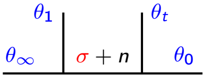


- block integrable kernels
- Widom's constants

- summation over partitions/Young diagrams

General solution of PVI [Gamayun, Iorgov, OL, '12]:

PVI tau function is a Fourier transform of $c = 1$ Virasoro conformal block:

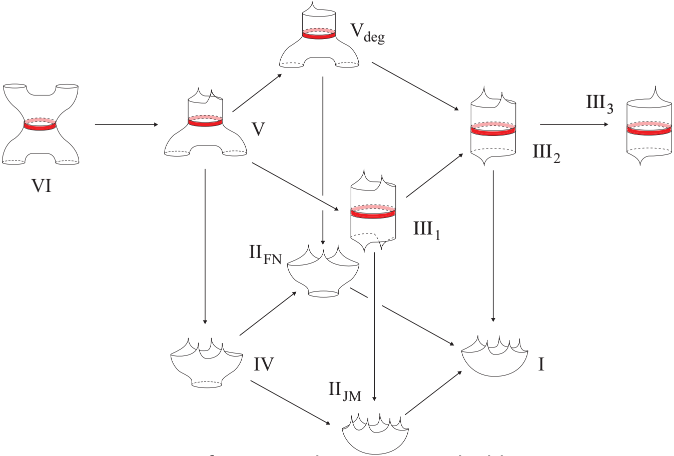
$$\tau(t) = \sum_{n\in\mathbb{Z}} e^{in\eta}\, \mathcal{B}(\vec{\theta}, \sigma + n, t) = \sum_{n\in\mathbb{Z}} e^{in\eta}$$  (t)

- $\mathcal{B}(\vec{\theta}, \sigma, t) = t^{\sigma^2 - \theta_0^2 - \theta_t^2} \sum_{k=0}^{\infty} B_k(\vec{\theta}, \sigma)\, t^k$
- B_k determined by commutation relations of Vir
- AGT correspondence [Alday, Gaiotto, Tachikawa, '09]:

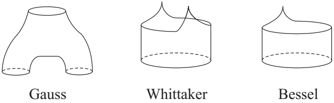
$$\mathcal{B}(t) = \mathcal{Z}_{\text{inst}}(t) = \begin{array}{c}\text{sum over pairs}\\ \text{of Young diagrams}\end{array}$$ [Nekrasov, '04]

Series representation for PVI tau function (proof in [Gavrylenko, OL, '16])

$$\boxed{\tau(t) = \sum_{n\in\mathbb{Z}} e^{in\eta} \sum_{\lambda,\mu\in\mathbb{Y}} \mathcal{B}_{\lambda,\mu}(\vec{\theta}, \sigma + n)\, t^{(\sigma+n)^2 + |\lambda| + |\mu|}}$$



- PVI, PV, $\mathrm{PIII}_{1,2,3}$ surfaces may be cut into solvable pieces

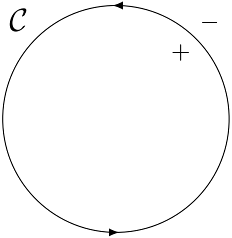


- More surprisingly, Fourier transform also appears in "irregular type" expansions for PI–PV at $t = \infty$.

Riemann-Hilbert setup

- let $\mathcal{C} \subset \mathbb{C}$ be a circle centered at the origin
- pick a loop $J(z) \in \mathrm{Hom}(\mathcal{C}, \mathrm{GL}_N(\mathbb{C}))$
- $J(z)$ continues into an annulus $\mathcal{A} \supset \mathcal{C}$

$$J(z) = \sum_{k \in \mathbb{Z}} J_k z^k,$$



Two Riemann-Hilbert problems:

$$\text{direct}: \quad J(z) = \Psi_-(z)^{-1} \Psi_+(z)$$

$$\text{dual}: \quad J(z) = \bar{\Psi}_+(z)\, \bar{\Psi}_-(z)^{-1}$$

Main definition: The tau function of RHPs defined by $(\mathcal{C}, J)$ is defined as Fredholm determinant

$$\tau[J] = \det_{H_+}\left(\Pi_+ J^{-1} \Pi_+ J \Pi_+\right),$$

where $H = L^2\left(\mathcal{C}, \mathbb{C}^N\right)$ and Π_+ is the orthogonal projection on H_+ along H_-.

Properties:

- dual RHP is solvable iff the operator $P := \Pi_+ J^{-1} \Pi_+$ is invertible on H_+, in which case $P^{-1} = \bar{\Psi}_+ \Pi_+ \bar{\Psi}_-^{-1} \Pi_+$
- likewise, for direct RHP and $Q := \Pi_+ J \Pi_+$, with $Q^{-1} = \Psi_+^{-1} \Pi_+ \Psi_- \Pi_+$
- if either direct or dual RHP is not solvable, then $\tau[J] = 0$
- $\tau[J]$ appears in the large size asymptotics of Toeplitz determinants with symbol J and is called Widom's constant in this context

If the direct RHP is solvable, then $\tau[J]$ may also be written as

$$\tau[J] = \det_H (\mathbf{1} + K)\,, \qquad K = \begin{pmatrix} 0 & \mathsf{a}_{+-} \\ \mathsf{a}_{-+} & 0 \end{pmatrix},$$

where $\mathsf{a}_{\pm\mp} = \Psi_\pm \Pi_\pm \Psi_\pm^{-1} - \Pi_\pm : H_\mp \to H_\pm$ are integral operators

$$(\mathsf{a}_{\pm\mp} f)(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \mathsf{a}_{\pm\mp}(z, z')\, f(z')\, dz',$$

with block integrable kernels

$$\mathsf{a}_{\pm\mp}(z, z') = \pm \frac{\mathbf{1} - \Psi_\pm(z)\, \Psi_\pm(z')^{-1}}{z - z'}.$$

In applications to Painlevé:

- $\Psi_\pm$ (direct factorization) are given and define the jump $J = \Psi_-{}^{-1}\Psi_+$
- $\Psi_\pm$ are expressed via classical special functions (Gauss, Kummer & Bessel for PVI, PV, PIII's)
- dual factorization ($\bar{\Psi}_\pm$ in $J = \bar{\Psi}_+ \bar{\Psi}_-^{-1}$) is the problem to be solved

Differentiation formula

Theorem: Let $(z,t) \mapsto J(z,t)$ be a smooth family of $\mathrm{GL}(N,\mathbb{C})$-loops which depend on an extra parameter t and admit direct & dual factorization. Then

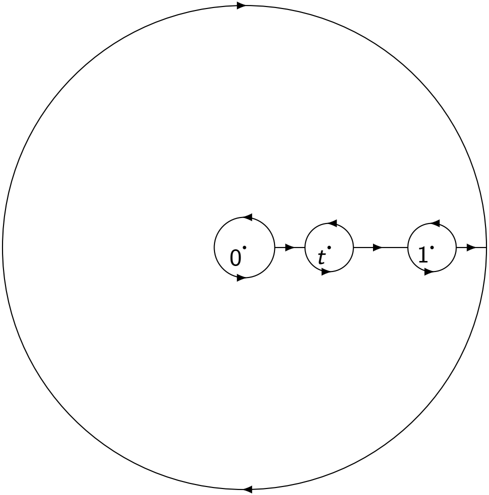
$$\partial_t \ln \tau[J] = \frac{1}{2\pi i}\oint_{\mathcal{C}} \mathrm{Tr}\left\{J^{-1}\partial_t J\left[\partial_z\bar{\Psi}_-\,\bar{\Psi}_-^{-1} + \Psi_+^{-1}\,\partial_z\Psi_+\right]\right\} dz.$$

- proof in [Widom, '74]; rediscovered by [Its, Jin, Korepin, '06]
- related results in the study of dependence of isomonodromic tau functions on monodromy [Bertola, '09]

Corollary: in isomonodromic RHPs,

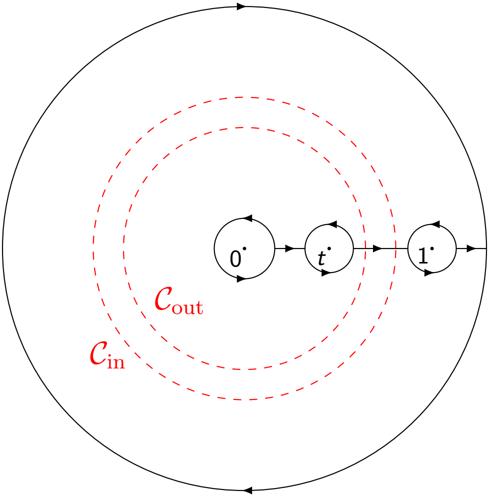
Widom's constant $\tau[J]$ $\simeq$ Jimbo-Miwa-Ueno tau function

Dual RHP_1 for $\tilde{\Psi}$



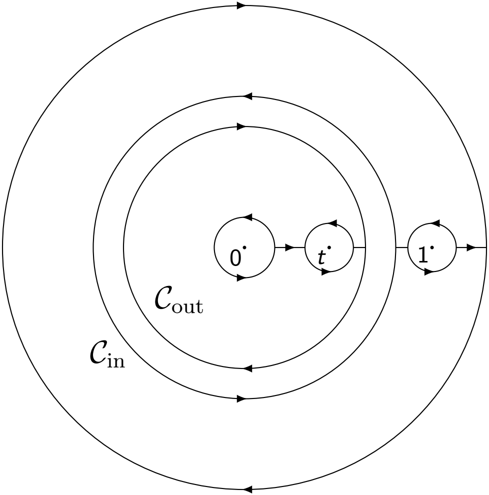
$$\tilde{\Psi}(z) = \begin{cases} G^{(\nu)}(z), & z \in D_\nu, \\ \Phi(z), & z \notin \mathbb{R}_{\geq 0} \cup \bar{D}_0 \cup \bar{D}_t \cup \bar{D}_1 \cup \bar{D}_\infty. \end{cases}$$

Dual RHP_1 for $\tilde{\Psi}$

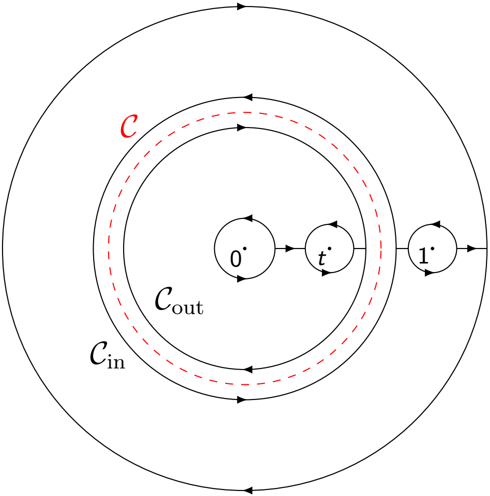


$$\hat{\Psi}(z) = \begin{cases} (-z)^{-\mathfrak{S}}\, \tilde{\Psi}(z)\,, & z \in \mathcal{A}, \\ \tilde{\Psi}(z)\,, & z \notin \bar{\mathcal{A}}. \end{cases}$$

Dual RHP_2 for $\hat{\Psi}$

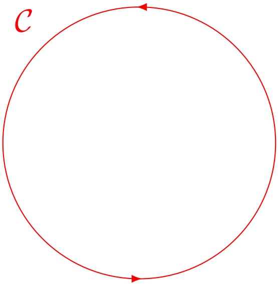


Dual RHP$_2$ for $\hat{\Psi}$



$$\bar{\Psi}(z) = \begin{cases} \Psi_+(z)^{-1}\hat{\Psi}(z), & \text{outside } \mathcal{C}, \\ \Psi_-(z)^{-1}\hat{\Psi}(z), & \text{inside } \mathcal{C}. \end{cases}$$

Dual RHP$_3$ for $\bar{\Psi}$



$$\bar{\Psi}(z) = \begin{cases} \Psi_+(z)^{-1}\hat{\Psi}(z), & \text{outside } \mathcal{C}, \\ \Psi_-(z)^{-1}\hat{\Psi}(z), & \text{inside } \mathcal{C}. \end{cases}$$

- contour $\mathcal{C}$ (single circle !), smooth jump $J : \mathcal{C} \to \mathrm{GL}(N, \mathbb{C})$ given by

$$J(z) = \Psi_-(z)^{-1}\Psi_+(z) = \bar{\Psi}_+(z)\,\bar{\Psi}_-(z)^{-1}$$

- we are in the previously described setup!

Widom's differentiation formula implies that

$$\partial_t \ln \tau[J] = \underbrace{\frac{\operatorname{Tr} A_0 A_t}{t} + \frac{\operatorname{Tr} A_t A_1}{t-1}}_{\partial_t \ln \tau_{\mathrm{JMU}}(t)} - \frac{\operatorname{Tr} A_0^+ A_t^+}{t},$$

so that in turn

$$\tau_{\mathrm{JMU}}(t) = t^{\frac{1}{2}\operatorname{Tr}\left(\mathfrak{S}^2 - \Theta_0^2 - \Theta_t^2\right)} \tau[J].$$

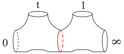
- Recall that

$$\tau[J] = \det(\mathbf{1} + K), \qquad K = \begin{pmatrix} 0 & \mathsf{a}_{+-} \\ \mathsf{a}_{-+} & 0 \end{pmatrix},$$

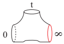
$$\mathsf{a}_{\pm\mp}(z, z') = \pm \frac{\mathbf{1} - \Psi_{\pm}(z)\,\Psi_{\pm}(z')^{-1}}{z - z'}.$$

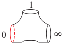
- $\tau_{\mathrm{JMU}}(t)$ for 4-point system written via auxiliary 3-point solutions
- hypergeometric representations for $N = 2 \Longrightarrow$ PVI tau function !

Schematically,

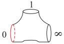
τ_{JMU} (

) $=$

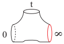
τ_{JMU} (

) τ_{JMU} (

) det (

$\mathbf{1}$ a_{+-} (

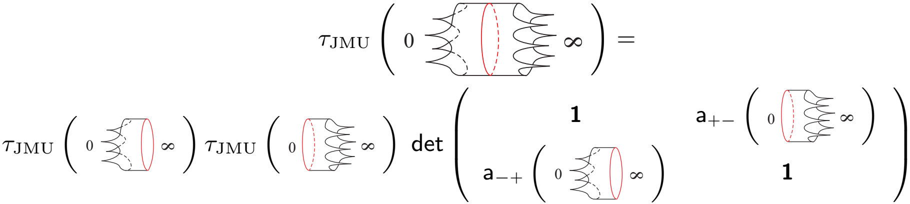
)

a_{-+} (

) $\mathbf{1}$

)

Similarly, for a linear system with 2 irregular singularities



Series representations

Given $K \in \mathbb{C}^{\mathfrak{X}\times\mathfrak{X}}$, we can expand Fredholm determinant

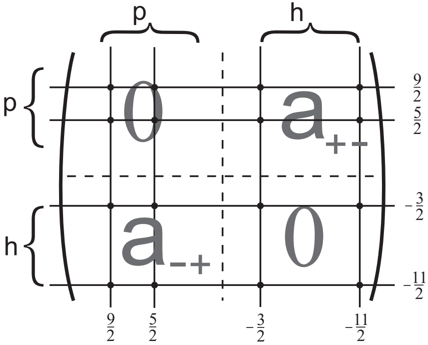
$$\det(\mathbf{1}+K) = \sum_{\mathfrak{Y}\in 2^{\mathfrak{X}}} \det K_{\mathfrak{Y}} = 1 + \sum_{m\in\mathfrak{X}} K_{mm} + \frac{1}{2!}\sum_{m,n\in\mathfrak{X}} \begin{vmatrix} K_{mm} & K_{mn} \\ K_{nm} & K_{nn} \end{vmatrix} + \ldots$$

- our case: $K = \begin{pmatrix} 0 & \mathsf{a}_{+-} \\ \mathsf{a}_{-+} & 0 \end{pmatrix}$
- in the Fourier basis,

  $$\mathsf{a}_{\pm\mp}(z,z') = \sum_{p,q\in\mathbb{Z}'_+} \mathsf{a}^{\pm p}_{\mp q} z^{-\frac{1}{2}\pm p} z'^{-\frac{1}{2}\pm q},$$

  with $\mathsf{a}^{\pm p}_{\mp q} \in \mathrm{Mat}_{N\times N}(\mathbb{C})$.
- multi-indices $m, n, \ldots$ of principal minors $\det K_{\mathfrak{Y}} = \det \begin{pmatrix} 0 & \mathsf{a}^{\mathsf{p}}_{\mathsf{h}} \\ \mathsf{a}^{\mathsf{h}}_{\mathsf{p}} & 0 \end{pmatrix}$ incorporate color indices $\alpha = 1, \ldots N$ and (half-)integer Fourier indices

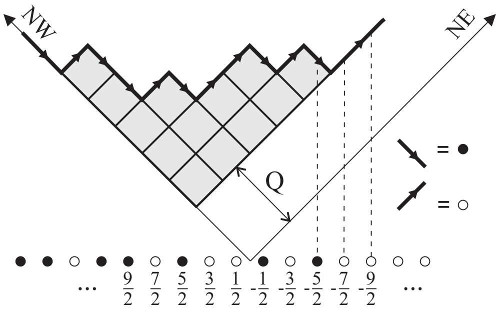
$N = 1$ **case**:



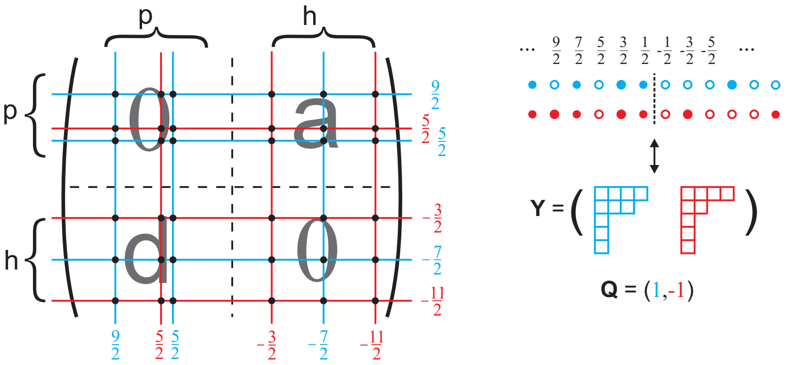
- combinatorial expansion

$$\det(\mathbf{1} + K) = \sum_{(\mathsf{p},\mathsf{h})} (-1)^{|\mathsf{p}|} \det \mathsf{a}_{\mathsf{h}}^{\mathsf{p}} \det \mathsf{a}_{\mathsf{p}}^{\mathsf{h}},$$

with balance condition $|\mathsf{p}| = |\mathsf{h}|$



- A Maya diagram is a map $\mathrm{m} : \mathbb{Z}' \to \{-1, 1\}$ subject to the condition $\mathrm{m}\,(p) = \pm 1$ for all but finitely many $p \in \mathbb{Z}'_{\pm}$ (positions of particles and holes)
- Maya diagram $=$ charged partition/Young diagram
- $\text{charge}(\mathrm{m}) = \sharp(\text{particles}) - \sharp(\text{holes})$
- for $N = 1$ principal minors are labeled by partitions



- balanced configurations (p,h) are in bijection with N-tuples of Young diagrams of zero total charge
- $\mathbb{M}_0^N \cong \mathbb{Y}^N \times \mathfrak{Q}_N$, where $\mathfrak{Q}_N$ denotes the A_{N-1} root lattice:

$$\mathfrak{Q}_N := \left\{ \vec{Q} \in \mathbb{Z}^N \,\middle|\, \sum_{\alpha=1}^{N} Q_\alpha = 0 \right\}.$$

- in the case $N = 2$, Widom's constant

$$\det(\mathbf{1}+K) = \sum_{(\lambda,\mu;n)\in\mathbb{Y}^2\times\mathbb{Z}} (-1)^{|\mathsf{p}|} \det(\mathsf{a}_{+-})_{\lambda,\mu,n} \det(\mathsf{a}_{-+})_{\lambda,\mu,n}$$

Isomonodromic examples

Explicit computation of elementary determinants $\det \mathsf{a}_{\mathsf{h}}^{\mathsf{p}}$, $\det \mathsf{a}_{\mathsf{p}}^{\mathsf{h}}$:

- a variant of Tracy-Widom conditions

$$\partial_z \Psi_\pm(z) = \Psi_\pm(z)\, A_\pm(z) + z^{-1} \Lambda_\pm(z)\, \Psi_\pm(z)\,,$$

  with $A_\pm(z)$ rational in z and $\Lambda_\pm(z)$ polynomial in $z^{\pm 1}$.

- acting with $\mathcal{L}_0 = z\partial_z + z'\partial_{z'} + 1$ e.g. on

$$\frac{\mathbf{1} - \Psi_+(z)\, \Psi_+(z')^{-1}}{z - z'} = \sum_{p,q \in \mathbb{Z}'_+} \mathsf{a}_{-q}^{\;p} z^{-\frac{1}{2}+p} z'^{-\frac{1}{2}+q}$$

  yields a system of linear matrix equations on Fourier modes $\mathsf{a}_{-q}^{\;p}$ thanks to the fact that $\mathcal{L}_0 \frac{1}{z-z'} = 0$.

- PVI, V, III semisimple cases $(N=2) \Longrightarrow$ Cauchy determinants

$$\det \frac{f_{p,\alpha}\, g_{q,\beta}}{p + q + \sigma_\alpha + \sigma_\beta}$$

Conclusions

Arbitrary RHPs:

1. A tau function (= Widom's constant) can be assigned to "any" RHP.
2. Given the direct factorization of the jump matrix, $\tau[J]$ may be written as a Fredholm determinant with a block integrable kernel.
3. Principal minor expansion of this determinant in the Fourier basis leads to combinatorial series over tuples of partitions.
4. Results can be generalized to many-oval contour (e.g. Garnier system)

Isomonodromic RHPs:

1. In RHPs of isomonodromic origin, $\tau[J] \simeq \tau_{\mathrm{JMU}}$, thanks to Widom's differentiation formula
2. Integral kernels and coefficients of combinatorial series can be computed explicitly when auxiliary solutions from the direct factorization have hypergeometric representations; in particular, for PVI, PV and PIIIs.

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THANK YOU!

Integrable kernel for $N=2$:

$$\mathsf{a}_{+-}(z,z') = \frac{(1-z')^{2\theta_1}\begin{pmatrix} K_{++}(z) & K_{+-}(z) \\ K_{-+}(z) & K_{--}(z)\end{pmatrix}\begin{pmatrix} K_{--}(z') & -K_{+-}(z') \\ -K_{-+}(z') & K_{++}(z')\end{pmatrix} - \mathbf{1}}{z-z'},$$

$$\mathsf{a}_{-+}(z,z') = \frac{\mathbf{1} - \left(1-\frac{t}{z'}\right)^{2\theta_t}\begin{pmatrix} \bar{K}_{++}(z) & \bar{K}_{+-}(z) \\ \bar{K}_{-+}(z) & \bar{K}_{--}(z)\end{pmatrix}\begin{pmatrix} \bar{K}_{--}(z') & -\bar{K}_{+-}(z') \\ -\bar{K}_{-+}(z') & \bar{K}_{++}(z')\end{pmatrix}}{z-z'}$$

with

$$K_{\pm\pm}(z) = {}_2F_1\left[\begin{matrix}\theta_1+\theta_\infty\pm\sigma, \theta_1-\theta_\infty\pm\sigma \\ \pm 2\sigma\end{matrix}; z\right],$$

$$K_{\pm\mp}(z) = \pm\frac{\theta_\infty^2-(\theta_1\pm\sigma)^2}{2\sigma(1\pm 2\sigma)}\, z\, {}_2F_1\left[\begin{matrix}1+\theta_1+\theta_\infty\pm\sigma, 1+\theta_1-\theta_\infty\pm\sigma \\ 2\pm 2\sigma\end{matrix}; z\right],$$

$$\bar{K}_{\pm\pm}(z) = {}_2F_1\left[\begin{matrix}\theta_t+\theta_0\mp\sigma, \theta_t-\theta_0\mp\sigma \\ \mp 2\sigma\end{matrix}; \frac{t}{z}\right],$$

$$\bar{K}_{\pm\mp}(z) = \mp t^{\mp 2\sigma} e^{\mp i\eta}\frac{\theta_0^2-(\theta_t\mp\sigma)^2}{2\sigma(1\mp 2\sigma)}\,\frac{t}{z}\, {}_2F_1\left[\begin{matrix}1+\theta_t+\theta_0\mp\sigma, 1+\theta_t-\theta_0\mp\sigma \\ 2\mp 2\sigma\end{matrix}; \frac{t}{z}\right].$$

Theorem [Gavrylenko, OL, '16]

The series expansion of Painlevé VI tau function around $t = 0$ is given by

$$\tau(t) = \sum_{n\in\mathbb{Z}} e^{in\eta} \mathcal{B}(\vec{\theta}, \sigma + n; t),$$

where the function $\mathcal{B}(\vec{\theta}, \sigma; t)$ is explicitly given by

$$\mathcal{B}\left(\vec{\theta}, \sigma; t\right) = N^{\theta_1}_{\theta_\infty,\sigma} N^{\theta_t}_{\sigma,\theta_0} t^{\sigma^2-\theta_0^2-\theta_t^2} (1-t)^{2\theta_t\theta_1} \sum_{\lambda,\mu\in\mathbb{Y}} \mathcal{B}_{\lambda,\mu}\left(\vec{\theta}, \sigma\right) t^{|\lambda|+|\mu|},$$

$$\mathcal{B}_{\lambda,\mu}(\boldsymbol{\theta}, \sigma) = \prod_{(i,j)\in\lambda} \frac{\left((\theta_t+\sigma+i-j)^2-\theta_0^2\right)\left((\theta_1+\sigma+i-j)^2-\theta_\infty^2\right)}{h_\lambda^2(i,j)\left(\lambda'_j-i+\mu_i-j+1+2\sigma\right)^2} \times$$

$$\times \prod_{(i,j)\in\mu} \frac{\left((\theta_t-\sigma+i-j)^2-\theta_0^2\right)\left((\theta_1-\sigma+i-j)^2-\theta_\infty^2\right)}{h_\mu^2(i,j)\left(\mu'_j-i+\lambda_i-j+1-2\sigma\right)^2},$$

$$N^{\theta_2}_{\theta_3,\theta_1} = \frac{\prod_{\epsilon=\pm} G\left(1+\theta_3+\epsilon(\theta_1+\theta_2)\right) G\left(1-\theta_3+\epsilon(\theta_1-\theta_2)\right)}{G(1-2\theta_1)G(1-2\theta_2)G(1+2\theta_3)}.$$