Invariant measures for NLS equations as limit of many-body quantum states

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I. Hartree theory

Energy: the **Hartree functional** is given by

$$\mathcal{E}_{\mathsf{H}}(\phi) = \int \left[|\nabla\phi(x)|^2 + v(x)|\phi(x)|^2 \right] dx + \frac{1}{2}\int w(x-y)|\phi(x)|^2|\phi(y)|^2\, dxdy$$

and acts on $L^2(\mathbb{R}^d)$ (we will consider $d = 1, 2, 3$).

We assume v is **confining** and $w \in L^\infty(\mathbb{R}^d)$ **pointwise non-negative** if $d = 1$ or of **positive type** if $d = 2, 3$.

Evolution: the **time-dependent Hartree equation** is given by

$$i\partial_t\phi_t = [-\Delta + v(x)]\, \phi_t + (w * |\phi_t|^2)\phi_t$$

Invariant measure: formally given by

$$d\mu_H = \frac{1}{Z} e^{-\left[\mathcal{E}_{\mathrm{H}}(\phi)+\kappa\|\phi\|_2^2\right]}\, d\phi$$

Constructive QFT in '70s: Nelson, Glimm-Jaffe, Simon, ...

Recently, problem awoke interest of dispersive pde's community.

Important application of this line of research is the **almost sure well-posedness** for **rough** initial data.

Results by: Lebowitz-Rose-Speer, Bourgain, Zhidkov, Bourgain-Bulut, Burq-Tzvetkov, Burq-Thomann-Tzvetkov, Nahmod-Oh-Rey-Bellet-Sheffield-Staffilani, Oh-Popovnicu, Oh-Quastel, Deng-Tzvetkov-Visciglia, Oh-Tzvetkov-Wang, Cacciafesta-de Suzzoni, Genovese-Lucá-Valeri, ...

Free functional: let

$$\mathcal{E}_0(\phi) = \int \left[|\nabla\phi(x)|^2 + v(x)|\phi(x)|^2 + \kappa|\phi(x)|^2 \right] dx = \langle \phi, h\phi \rangle$$

with

$$h = -\Delta + v(x) + \kappa = \sum_n \lambda_n |u_n\rangle\langle u_n|$$

We **assume**

$$\begin{cases} \mathrm{Tr}\, h^{-1} & = \sum_{n\in\mathbb{N}} \lambda_n^{-1} < \infty \qquad \text{for } d = 1 \\ \mathrm{Tr}\, h^{-2} & = \sum_{n\in\mathbb{N}} \lambda_n^{-2} < \infty \qquad \text{for } d = 2, 3 \end{cases}$$

Free measure: to define $d\mu_0 \sim \exp(-\mathcal{E}_0(\phi))d\phi$, we **expand**

$$\phi(x) = \sum_{n\in\mathbb{N}} \frac{\omega_n}{\sqrt{\lambda_n}} u_n(x) \qquad \Rightarrow \qquad \mathcal{E}_0(\phi) = \langle \phi, h\phi \rangle = \sum_{n\in\mathbb{N}} |\omega_n|^2$$

Hence we define μ_0 on $\mathbb{C}^{\mathbb{N}} = \{\{\omega_n\}_{n\in\mathbb{N}} : \omega_n \in \mathbb{C}\}$ as **product of iid Gaussian measures** with densities

$$\frac{1}{\pi} e^{-|\omega_n|^2} d\omega_n d\omega_n^*$$

Expected L^2 norm: observe that

$$\mathbb{E}_{\mu_0} \|\phi\|_2^2 = \mathbb{E}_{\mu_0} \sum_{n \in \mathbb{N}} \frac{|\omega_n|^2}{\lambda_n} = \sum_{n \in \mathbb{N}} \frac{1}{\lambda_n} = \mathrm{Tr}\, h^{-1}$$

is finite for $d = 1$, but it is **infinite** for $d = 2, 3$.

Hartree invariant measure: for $d = 1$, we can define

$$\mu_H = \frac{1}{Z} e^{-W} \mu_0$$

with interaction

$$W(\phi) = \frac{1}{2} \int w(x-y) |\phi(x)|^2 |\phi(y)|^2 dx dy \leq \frac{\|w\|_\infty}{2} \|\phi\|_2^4$$

For $d = 2, 3$, on the other hand, $W = \infty$ **almost surely**.

Wick ordering: for $K > 0$ we introduce **cutoff fields**

$$\phi_K(x) = \sum_{n \leq K} \frac{\omega_n}{\sqrt{\lambda_n}} u_n(x)$$

and we define

$$\rho_K(x) = \mathbb{E}_{\mu_0} |\phi_K(x)|^2 = \sum_{n \leq K} \lambda_n^{-1} \, |u_n(x)|^2$$

and the **cutoff renormalized interaction**

$$W_K = \frac{1}{2} \int w(x-y) \left[|\phi_K(x)|^2 - \rho_K(x) \right] \left[|\phi_K(y)|^2 - \rho_K(y) \right] dx dy$$

Lemma: W_K is **Cauchy sequence** in $L^p(\mathbb{C}^{\mathbb{N}}, d\mu_0)$ for all $p < \infty$. We denote by W^r its limit (independent of p).

For $d = 2, 3$, we define renormalized Gibbs measure

$$\mu_H^r = \frac{1}{\int e^{-W^r(\phi)} d\mu_0(\phi)} e^{-W^r} \mu_0$$

Note that μ_H^r is **invariant** with respect to the Hartree flow.

II. Mean field quantum systems

Hamilton operator: has form

$$H_N = \sum_{j=1}^{N} \left[-\Delta_{x_j} + v(x_j)\right] + \frac{1}{N}\sum_{i<j}^{N} w(x_i - x_j) \quad \text{on } L^2_s(\mathbb{R}^{dN})$$

Ground state: $\psi_N \simeq \phi_0^{\otimes N}$, where ϕ_0 is **minimizer** of $\mathcal{E}_H$.

Dynamics: governed by the many-body **Schrödinger** equation

$$i\partial_t \psi_{N,t} = H_N \psi_{N,t}$$

Convergence to Hartree: if $\psi_{N,0} \simeq \phi^{\otimes N}$, then $\psi_{N,t} \simeq \phi_t^{\otimes N}$ where ϕ_t solves **time-dependent Hartree equation**.

Rigorous works: Hepp, Ginibre-Velo, Spohn, Erdős-Yau, Bardos-Golse-Mauser, Fröhlich-Knowles-Schwarz, Rodnianski-S., Knowles-Pickl, Fröhlich-Knowles-Pizzo, Grillakis-Machedon-Margetis, T.Chen-Pavlovic, X.Chen-Holmer, Ammari-Nier, Lewin-Nam-S., . . .

Question: what corresponds to **Hartree invariant measure** in many-body setting?

Thermal equilibrium: at **temperature** β^{-1}, it is described by

$$\mathbb{E}_\beta A = \mathrm{Tr}\, A \varrho_\beta$$

with density matrix

$$\varrho_\beta = \frac{1}{Z_\beta} e^{-\beta H_N}, \qquad Z_\beta = \mathrm{Tr}\, e^{-\beta H_N}$$

Remark 1: if $\beta > 0$ fixed, ϱ_β still exhibits **condensation**. At one-particle level this leads to **trivial measure** δ_{ϕ_0}.

To recover invariant measure, need to take $\beta = 1/N$.

Remark 2: number of particles at many-body level corresponds to L^2-norm at Hartree level.

To recover invariant measure, need **fluctuations** of number of particles.

III. Fock space and grand canonical ensemble

Fock space: we define

$$\mathcal{F} = \bigoplus_{m \geq 0} L^2(\mathbb{R}^d)^{\otimes_s m} = \bigoplus_{m \geq 0} L^2_s(\mathbb{R}^{md})$$

Creation and annihilation operators: for $f \in L^2(\mathbb{R}^d)$, let

$$(a^*(f)\Psi)^{(m)}(x_1, \dots, x_m) = \frac{1}{\sqrt{m}} \sum_{j=1}^{m} f(x_j)\Psi^{(m-1)}(x_1, \dots, \not{x}_j, \dots, x_m)$$

$$(a(f)\Psi)^{(m)}(x_1, \dots, x_m) = \sqrt{m+1} \int dx\, \overline{f}(x)\Psi^{(m+1)}(x, x_1, \dots, x_m)$$

They satisfy **canonical commutation relations**

$$[a(f), a^*(g)] = \langle f, g \rangle, \quad [a(f), a(g)] = [a^*(f), a^*(g)] = 0$$

We define operator valued **distributions** $a(x), a^*(x)$ such that

$$a^*(f) = \int f(x)\, a^*(x)\, dx, \quad \text{and } a(f) = \int \overline{f(x)}\, a(x)\, dx$$

Number of particles operator: is given by

$$\mathcal{N} = \int a^*(x) a(x)\, dx$$

Hamilton operator: is defined through

$$\mathcal{H}_N = \int a^*(x)\, [-\Delta_x + v(x)]\, a(x) + \frac{1}{2N} \int w(x-y) a^*(x) a^*(y) a(y) a(x)$$

Notice that $[\mathcal{H}_N, \mathcal{N}] = 0$ and

$$\mathcal{H}_N|_{\mathcal{F}_m} = \sum_{j=1}^{m} \left[-\Delta_{x_j} + v(x_j) \right] + \frac{1}{N} \sum_{i<j}^{m} w(x_i - x_j)$$

Grand canonical ensemble: at inverse **temperature** $\beta = N^{-1}$ and **chemical potential** κ, equilibrium is described by

$$\varrho_N = \frac{1}{Z_N} e^{-\frac{1}{N}(\mathcal{H}_N + \kappa \mathcal{N})}, \qquad \text{with} \qquad Z_N = \text{Tr}\, e^{-\frac{1}{N}(\mathcal{H}_N + \kappa \mathcal{N})}$$

Rescaled operators: it is useful to define

$$a_N(x) = \frac{1}{\sqrt{N}} a(x), \qquad a_N^*(x) = \frac{1}{\sqrt{N}} a^*(x)$$

Expressed in terms of the **rescaled fields**, we find

$$\varrho_N = Z_N^{-1} \exp \Big[- \int a_N^*(x)(-\Delta_x + v(x) + \kappa) a_N(x)\, dx + \frac{1}{2} \int w(x-y)\, a_N^*(x)\, a_N^*(y)\, a_N(y)\, a_N(x)\, dx dy \Big]$$

Notice that

$$[a_N(x), a_N^*(y)] = \frac{1}{N}\delta(x-y), \qquad [a_N(x), a_N(y)] = [a_N^*(x), a_N^*(y)] = 0$$

are **almost commuting** operators.

IV. Non-interacting Gibbs states and Wick ordering

Non-interacting Gibbs state: we **diagonalize**

$$\int a_N^*(x)\left[-\Delta_{x_j}+v(x_j)+\kappa\right]a_N(x)\,dx=\sum_j \lambda_j a_N^*(u_j)a_N(u_j)$$

which leads to

$$\varrho_N^{(0)}=\frac{1}{Z_N^{(0)}}\,e^{-\sum_j \lambda_j a_N^*(u_j)a_N(u_j)}$$

Expectation of rescaled **number of particles**

$$\mathbb{E}_N^{(0)}\,a_N^*(u_i)a_N(u_i)=\frac{\mathrm{Tr}\,a_N^*(u_i)a_N(u_i)\,e^{-\lambda_i a_N^*(u_i)a_N(u_i)}}{\mathrm{Tr}e^{-\lambda_i a_N^*(u_i)a_N(u_i)}}=\frac{1}{N}\frac{1}{e^{\lambda_i/N}-1}$$

Hence

$$\mathbb{E}_N^{(0)}\frac{1}{N}\sum_i a_N^*(u_i)a_N(u_i)=\frac{1}{N}\sum_{i\in\mathbb{N}}\frac{1}{e^{\lambda_i/N}-1}=\begin{cases}O(1), & \text{for } d=1\\ \to\infty, & \text{for } d=2,3\end{cases}$$

Interaction: expectation of

$$W_N = \frac{1}{2}\int w(x-y)a_N^*(x)a_N^*(y)a_N(y)a_N(x)dxdy$$

is **finite** but, for $d = 2, 3$, it diverges, as $N \to \infty$.

Wick ordering: replace W_N with the Wick ordered interaction

$$W_N^r = \frac{1}{2}\int w(x-y)\left[a_N^*(x)a_N(x) - \rho_N(x)\right]\left[a_N^*(y)a_N(y) - \rho_N(y)\right]dxdy$$

with

$$\rho_N(x) = \mathbb{E}_N^{(0)}\, a_N^*(x)a_N(x) = \frac{1}{N}\sum_{j\in\mathbb{N}}\frac{|u_j(x)|^2}{e^{\lambda_j/N}-1}$$

We write the resulting **grand canonical** density matrix

$$\varrho_N^r = \frac{1}{Z_N^r}e^{-\mathcal{H}_N^r} = \frac{1}{Z_N^r}e^{-(\mathcal{H}_{N,0}+W_N^r)}$$

with

$$\mathcal{H}_{N,0} = \int a_N^*(x)\left[-\Delta_x + v(x) + \kappa\right]a_N(x)\,dx$$

V. Comparison with invariant measure for Hartree

Correlation functions: for $k \in \mathbb{N}$, define correlation function $\gamma_N^{(k)}$ as non-negative trace class operator on $L^2(\mathbb{R}^{kd})$ with **kernel**

$$\begin{aligned}\gamma_N^{(k)}(x_1,\dots,x_k;y_1,\dots,y_k) \\ &= \mathbb{E}_N^r\, a_N^*(x_1)\dots a_N^*(x_k)a_N(y_k)\dots a_N(y_1) \\ &= \mathrm{Tr}\, a_N^*(x_1)\dots a_N^*(x_k)a_N(y_k)\dots a_N(y_1)\,\varrho_N^r\end{aligned}$$

Joint moments: define $\gamma_H^{(k)}$ of **invariant measure** through

$$\begin{aligned}\gamma_H^{(k)}(x_1,\dots,x_k;y_1,\dots,y_k) \\ &= \mathbb{E}_H^r\, \overline{\phi}(x_1)\dots\overline{\phi}(x_k)\phi(y_k)\dots\phi(y_1) \\ &= \frac{\int \overline{\phi}(x_1)\dots\overline{\phi}(x_k)\phi(y_k)\dots\phi(y_1)\, e^{-W^r(\phi)}d\mu_0(\phi)}{\int e^{-W^r(\phi)}d\mu_0(\phi)}\end{aligned}$$

Conjecture: we expect that, for all fixed $k \in \mathbb{N}$,

$$\lim_{N\to\infty} \left\| \gamma_N^{(k)} - \gamma_H^{(k)} \right\|_{\mathrm{HS}} = 0$$

Theorem [Lewin-Nam-Rougerie, 2016]: let $d = 1$. Then conjecture holds true, with no need for renormalization.

In **[Fröhlich-Knowles-S.-Sohinger, 2017]** we give different proof of this theorem.

Very recently, **[Lewin-Nam-Rougerie, 2018]** announced proof of conjecture for $d = 2$ (renormalization needed).

In most interesting case $d = 3$, conjecture remains open. We prove it, but only for **slightly modified** many-body Gibbs states.

Modification: for fixed $\eta > 0$, we consider

$$\varrho^r_{N,\eta} = \frac{1}{Z^r_{N,\eta}}\, e^{-\eta \mathcal{H}_{N,0}}\, e^{-[(1-2\eta)\mathcal{H}_{N,0} + W^r_N]}\, e^{-\eta \mathcal{H}_{N,0}}$$

We denote by $\gamma^{(k)}_{\eta,N}$ the correlation functions associated to $\varrho^r_{N,\eta}$.

Remark: $\varrho^r_{N,\eta}$ is still **density matrix** of a quantum state.

Theorem [Fröhlich-Knowles-S.-Sohinger, 2017]: let
$d = 2, 3$,

$$h = -\Delta + v(x) + \kappa$$

with $\mathrm{Tr}\, h^{-2} < \infty$, $w \in L^\infty(\mathbb{R}^d)$ positive definite. Then, for all fixed $\eta > 0$ and $k \in \mathbb{N}$, we have

$$\lim_{N\to\infty} \left\| \gamma^{(k)}_{N,\eta} - \gamma^{(k)}_H \right\|_{\text{HS}} = 0$$

VI. Time dependent correlations (for $d = 1$)

Observables: for $\xi \in \mathcal{L}(L^2(\mathbb{R}^k))$, we define **random variable**

$$\Theta(\xi) = \int dx_1 \ldots dx_k dy_1 \ldots dy_k \, \xi(x_1, \ldots, x_k; y_1, \ldots, y_k) \times \overline{\phi}(x_1) \ldots \overline{\phi}(x_k) \phi(y_k) \ldots \phi(y_1)$$

and **quantum observable** (on $\mathcal{F}$)

$$\Theta_N(\xi) = \int dx_1 \ldots dx_k dy_1 \ldots dy_k \, \xi(x_1, \ldots, x_k; y_1, \ldots, y_k) \times a_N^*(x_1) \ldots a_N^*(x_k) a_N(y_k) \ldots a_N(y_1)$$

Dynamics: let S_t be **nonlinear Hartree flow**. We define

$$\Psi^t \left[\Theta(\xi)\right] = \int dx_1 \ldots dx_k dy_1 \ldots dy_k \, \xi(x_1, \ldots, x_k; y_1, \ldots, y_k) \times \overline{S_t \phi}(x_1) \ldots \overline{S_t \phi}(x_k) S_t \phi(y_k) \ldots S_t \phi(y_1)$$

and **quantum evolution**

$$\Psi_N^t \left[\Theta_N(\xi)\right] = \int dx_1 \ldots dx_k dy_1 \ldots dy_k \, \xi(x_1, \ldots, x_k; y_1, \ldots, y_k) \times e^{-i\mathcal{H}_N t} a_N^*(x_1) \ldots a_N^*(x_k) a_N(y_k) \ldots a_N(y_1) e^{i\mathcal{H}_N t}$$

Theorem [Fröhlich-Knowles-S.-Sohinger, 2018]: Let $w \in L^\infty(\mathbb{R})$, non-negative. Given $k \in \mathbb{N}$, $\xi_j \in \mathcal{L}(L^2(\mathbb{R}^{p_j}))$ and times t_j, for $j = 1, \ldots, k$, we have

$$\mathbb{E}_N \Psi_N^{t_1}[\Theta_N(\xi_1)] \ldots \Psi_N^{t_k}[\Theta_N(\xi_k)] \\ \to \mathbb{E}_H \, \Psi^{t_1}[\Theta(\xi_1)] \ldots \Psi^{t_k}[\Theta(\xi_k)]$$

as $N \to \infty$.

Remark: taking $k = 1$ and using **invariance of quantum state**, Theorem implies in particular **invariance of nonlinear Gibbs measure** w.r.t. the Hartree flow.

VII. Some ideas from the proof

Duhamel expansion: start from

$$e^{-[(1-2\eta)\mathcal{H}_{N,0}+W_N^r]}$$

$$= e^{-(1-2\eta)\left[\mathcal{H}_{N,0}+\frac{1}{1-2\eta}W_N^r\right]}$$

$$= e^{-(1-2\eta)\mathcal{H}_{N,0}} + \frac{1}{1-2\eta}\int_0^{1-2\eta} dt\, e^{-(1-2\eta-t)\mathcal{H}_{N,0}}\, W_N^r e^{-t\left[\mathcal{H}_{N,0}+\frac{1}{1-2\eta}W_N^r\right]}$$

Iterating, we find

$$e^{-\eta\mathcal{H}_{N,0}}e^{-(1-2\eta)\mathcal{H}_N}e^{-\eta\mathcal{H}_{N,0}}$$

$$= e^{-\mathcal{H}_{N,0}} + \sum_{m=1}^{n-1}\frac{1}{(1-2\eta)^m}\int_\eta^{1-\eta} dt_1 \cdots \int_\eta^{t_{m-1}} dt_m$$

$$\times\, e^{-(1-t_1)\mathcal{H}_{N,0}}\, W_N^r e^{-(t_1-t_2)\mathcal{H}_{N,0}} W_N^r \dots W_N^r\, e^{-t_m\mathcal{H}_{N,0}}$$

$$+ \frac{1}{(1-2\eta)^n}\int_\eta^{1-\eta} dt_1 \cdots \int_\eta^{t_{n-1}} dt_n$$

$$\times\, e^{-(1-t_1)\mathcal{H}_{N,0}}\, W_N^r \dots W_N^r e^{-(t_n-\eta)\left[\mathcal{H}_{N,0}+\frac{1}{1-2\eta}W_N^r\right]} e^{-\eta\mathcal{H}_{N,0}}$$

Evolved fields operator: remark that

$$e^{t\mathcal{H}_{0,N}} a_N^*(f) e^{-t\mathcal{H}_{0,N}} = a_N^*(e^{-th/N} f)$$

Fully expanded terms: need to compute **free expectations**!

Wick theorem: we have

$$\mathbb{E}_N^{(0)}\, a_N^{\sharp_1}(f_1) \ldots a_N^{\sharp_{2m}}(f_{2m})$$
$$= \sum_\pi \mathbb{E}_N^{(0)} \left[a_N^{\sharp_{i_1}}(f_{i_1}) a_N^{\sharp_{\ell_1}}(f_{\ell_1}) \right] \ldots \mathbb{E}_N^{(0)} \left[a_N^{\sharp_{i_m}}(f_{i_m}) a_N^{\sharp_{\ell_m}}(f_{\ell_m}) \right]$$

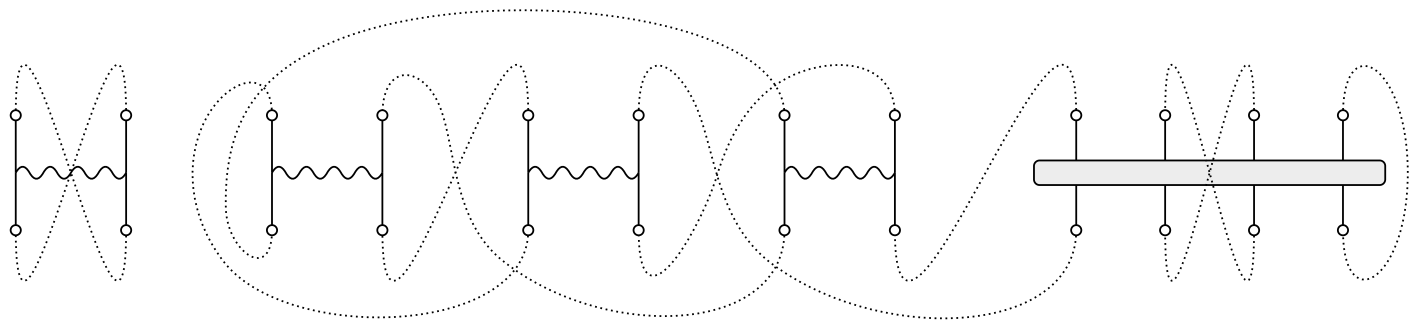
Non-vanishing expectations: are only

$$\mathbb{E}_{N,\kappa}^{(0)}\, [a_N^*(x) a_N(y)] = \frac{1}{N} \frac{1}{e^{h/N} - 1}(x;y)$$
$$\mathbb{E}_N^{(0)}\, [a_N(x) a_N^*(y)] = \frac{1}{N} \frac{1}{e^{h/N} - 1}(x;y) + \frac{1}{N} \delta(x-y)$$

Diagrammatic expansion: recall that

$$W_N^r = \frac{1}{2}\int w(x-y)\,[a_N^*(x)a_N(x) - \rho_N(x)]\,[a_N^*(y)a_N(y) - \rho_N(y)]\,dxdy$$

Pairings are encoded in **Feynman diagrams**



Bound: using diagrammatic representation and **assumption**

$$\operatorname{Tr} h^{-2} < \infty,$$

we conclude that each pairing is bounded, **uniformly** in N.

Convergence: as $N \to \infty$, each pairing tends to corresponding term in expansion of **Hartree invariant measure**.

Error term: use **Cauchy-Schwarz** to get rid of interacting term.

Here, for $d = 2, 3$, we need **modification** to avoid interacting exponential carrying full time.

Final obstacle: number of pairing $\sim (2n)!$, time integral $\sim 1/n!$

Hence, series **does not converge**!

Borel resummation: given **formal** power series representation

$$A(z) = \sum_{m \geq 0} a_m z^m$$

of **analytic** A, define

$$B(z) = \sum_{m \geq 0} \frac{a_m}{m!} z^m$$

Formally, we can then **reconstruct** A through

$$A(z) = \int_0^\infty e^{-t} B(tz) dt$$

Theorem [Sokal, 1980]: Let $A(z)$ and $(A_N(z))_{N\in\mathbb{N}}$ be **analytic** on ball

$$\mathcal{C}_R = \left\{z \in \mathbb{C} : (\mathrm{Re}\, z - R)^2 + \mathrm{Im}^2 z \leq R^2\right\}$$

for some $R > 0$. For $n \in \mathbb{N}$ suppose

$$A(z) = \sum_{m=0}^{n-1} a_m z^m + R_n(z), \qquad A_N(z) = \sum_{m=0}^{n-1} a_{m,N} z^m + R_{n,N}(z)$$

with

$$|a_m| + \sup_N |a_{m,N}| \leq C^m m!, \qquad |R_m(z)| + \sup_N |R_{m,N}(z)| \leq C^m |z|^m\, m!$$

for all $m \in \mathbb{N}$, $z \in \mathcal{C}_R$.

Suppose moreover that, for all $m \in \mathbb{N}$: $\lim_{N\to\infty} a_{m,N} = a_m$.

Then $A_N(z) \to A(z)$ for all $z \in \mathcal{C}_R$.

VIII. Appendix: the counterterm problem

Wick-ordering of many-body Hamiltonian: given

$$\mathcal{H}_N = \int a_N^*(x)\,[-\Delta_x + v(x) + \kappa]\,a_N(x)\,dx + \frac{1}{2}\int w(x-y)a_N^*(x)a_N(x)a_N^*(y)a_N(y)\,dxdy$$

we rewrite it as

$$\mathcal{H}_N = \int a_N^*(x)\,[-\Delta_x + v(x) + (w*\rho_N)(x) + \kappa]\,a_N(x)\,dx - \langle w*\rho_N, \rho_N\rangle + \frac{1}{2}\int w(x-y)\,[a_N^*(x)a_N(x) - \rho_N(x)]\,[a_N^*(y)a_N(y) - \rho_N(x)]\,dxdy$$

Subtracting constant and **shifting** chemical potential, we obtain

$$\widetilde{\mathcal{H}}_N = \int a_N^*(x)\,[-\Delta_x + v(x) + (w*(\rho_N - \bar{\rho}_N))(x) + \kappa]\,a_N(x)dx + \frac{1}{2}\int w(x-y)\,[a_N^*(x)a_N(x) - \rho_N(x)]\,[a_N^*(y)a_N(y) - \rho_N(x)]\,dxdy$$

with $\bar{\rho}_N = \mathbb{E}^{(0)}_{-\Delta+\kappa}\,a_N^*(x)a_N(x)$ **independent** of x.

Fix point problem: theorem can be applied to $\widetilde{\mathcal{H}}_N$ if we find

$$\widetilde{v} = v + (w * (\rho_N - \bar{\rho}_N)) \quad \text{s.t.} \qquad \rho_N(x) = \mathbb{E}^{(0)}_{-\Delta+\widetilde{v}+\kappa} a_N^*(x) a_N(x)$$

Theorem [Fröhlich-Knowles-S.-Sohinger, 2017]: Let $v \geq 0$ such that $v(x+y) \leq Cv(x)v(y)$ and

$$\mathrm{Tr}\,(-\Delta + v + \kappa)^{-2} < \infty.$$

Then for every $N \in \mathbb{N}$ there exists $\widetilde{v}_N$ solving the **counterterm problem**. Furthermore there is a **limiting potential** $\widetilde{v}$ such that

$$\lim_{N\to\infty} \left\| (-\Delta + \widetilde{v}_N + \kappa)^{-1} - (-\Delta + \widetilde{v} + \kappa)^{-1} \right\|_{\mathrm{HS}} = 0$$

Hence, after a **change of the chemical potential**, modified many-body quantum Gibbs state associated with $\mathcal{H}_N$ is s.t.

$$\lim_{N\to\infty} \left\| \gamma^{(k)}_{N,\eta} - \gamma^{(k)}_H \right\|_{\mathrm{HS}} = 0$$

where $\gamma^{(k)}_H$ are moments of **Hartree invariant measure** with external potential $\widetilde{v}$.