Topological invariants in disordered topological insulators — Subtitle: Spectral localizer of an index pairing

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ICMP, Montreal, July, 2018

Plan of the talk

- Winding number as prototype of an odd index pairing
- Construction of associated spectral localizer
- Main result: invariant as half-signature of spectral localizer
- Proof via spectral flow
- Even dimensional case
- Proof via fuzzy spheres
- Numerical results

Winding number as odd index pairing

For differentiable map

$$k \in \mathbb{R}/(2\pi\mathbb{Z}) = \mathbb{T}^1 \;\mapsto\; A(k) \in \mathbb{C}^{N\times N}$$

of invertible matrices, set

$$\mathrm{Wind}(A) \;=\; \frac{1}{2\pi i}\int_{\mathbb{T}^1} dk \;\mathrm{Tr}(A(k)^{-1}\partial_k A(k)) \;\in\; \mathbb{Z}$$

View A as multiplication operator on $L^2(\mathbb{T}^1)$

Theorem (Fritz Noether 1921, Gohberg-Krein 1960)

Let Π be Hardy projection onto $H^2 \subset L^2(\mathbb{T}^1)$
Then $\Pi A \Pi + (\mathbf{1} - \Pi)$ is Fredholm and:

$$\mathrm{Wind}(A) \;=\; \mathrm{Ind}\big(\Pi A \Pi + (\mathbf{1} - \Pi)\big)$$

Winding number in Fourier space

After Fourier $\mathcal{F} : L^2(\mathbb{T}^1) \to \ell^2(\mathbb{Z})$: convolution operator A

Differentiability of $A \;\cong\;$ bounded non-commutative derivative

$$\nabla A \;=\; i\,[D,A]$$

where D is unbounded position (dual Dirac) operator $D|n\rangle = n|n\rangle$

Theorem

Let $\Pi = (D > 0)$ be Hardy projection. Then

$$\mathrm{Wind}(A) \;=\; \mathrm{Ind}\big(\Pi A \Pi + (\mathbf{1} - \Pi)\big)$$

Physics: invariant for $1d$ disordered chiral topological insulators

Mathematically: canonical odd index paring of invertible A on $\mathcal{H}$ with an odd Fredholm module specified by a Dirac operator D with compact resolvent and bounded commutator $[D, A]$

New numerical technique: spectral localizer

For tuning parameter $\kappa > 0$ introduce spectral localizer:

$$L_\kappa = \begin{pmatrix} \kappa D & A \\ A^* & -\kappa D \end{pmatrix}$$

A_ρ restriction of A (Dirichlet b.c.) to range of $\chi(|D| \leq \rho)$

$$L_{\kappa,\rho} = \begin{pmatrix} \kappa D_\rho & A_\rho \\ A_\rho^* & -\kappa D_\rho \end{pmatrix}$$

Clearly selfadjoint matrix:

$$(L_{\kappa,\rho})^* = L_{\kappa,\rho}$$

Fact 1: $L_{\kappa,\rho}$ is gapped, namely $0 \notin L_{\kappa,\rho}$

Fact 2: $L_{\kappa,\rho}$ has spectral asymmetry measured by signature

Fact 3: signature linked to topological invariant

Theorem (with Loring 2017)

Given $D = D^*$ with compact resolvent and invertible A with invertibility gap $g = \|A^{-1}\|^{-1}$. Provided that

$$\|[D,A]\| \;\leq\; \frac{g^3}{12\,\|A\|\,\kappa} \qquad (*)$$

and

$$\frac{2\,g}{\kappa} \;\leq\; \rho \qquad (**)$$

the matrix $L_{\kappa,\rho}$ is invertible and with $\Pi = \chi(D \geq 0)$

$$\tfrac{1}{2}\,\mathrm{Sig}(L_{\kappa,\rho}) \;=\; \mathrm{Ind}\big(\Pi A \Pi + (\mathbf{1} - \Pi)\big)$$

How to use: form (*) infer κ, then ρ from (**)

If A unitary, $g = \|A\| = 1$ and $\kappa = (12\|[D,A]\|)^{-1}$ and $\rho = \frac{2}{\kappa}$

Hence **small** matrix of size $\leq$ 100 sufficient! Great for numerics!

Why it can work:

Proposition

If (*) and (**) hold,

$$L_{\kappa,\rho}^2 \;\geq\; \frac{g^2}{2}$$

Proof:

$$L_{\kappa,\rho}^2 \;=\; \begin{pmatrix} A_\rho A_\rho^* & 0 \\ 0 & A_\rho^* A_\rho \end{pmatrix} + \kappa^2 \begin{pmatrix} D_\rho^2 & 0 \\ 0 & D_\rho^2 \end{pmatrix} + \kappa \begin{pmatrix} 0 & [D_\rho, A_\rho] \\ [D_\rho, A_\rho]^* & 0 \end{pmatrix}$$

Last term is a perturbation controlled by (*)

First two terms positive (indeed: close to origin and away from it)

Now $A^*A \geq g^2$, but $(A^*A)_\rho \neq A_\rho^* A_\rho$

This issue can be dealt with by tapering argument:

Proposition (Bratelli-Robinson)

For $f:\mathbb{R}\to\mathbb{R}$ with Fourier transform defined without $\sqrt{2\pi}$,

$$\|[f(D),A]\| \;\leq\; \|\widehat{f'}\|_1\;\|[D,A]\|$$

Lemma

$\exists$ even function $f_\rho:\mathbb{R}\to[0,1]$ with $f_\rho(x)=0$ for $|x|\geq\rho$ and $f_\rho(x)=1$ for $|x|\leq\frac{\rho}{2}$ such that $\|\widehat{f_\rho'}\|_1=\frac{8}{\rho}$

With this, $f=f_\rho(D)=f_\rho(|D|)$ and $\mathbf{1}_\rho=\chi(|D|\leq\rho)$:

$$\begin{aligned}
A_\rho^*A_\rho \;&=\; \mathbf{1}_\rho A^*\mathbf{1}_\rho A\mathbf{1}_\rho \;\geq\; \mathbf{1}_\rho A^* f^2 A\mathbf{1}_\rho \\
&=\; \mathbf{1}_\rho fA^*Af\mathbf{1}_\rho \;+\; \mathbf{1}_\rho\big([A^*,f]fA+fA^*[f,A]\big)\mathbf{1}_\rho \\
&\geq\; g^2 f^2 \;+\; \mathbf{1}_\rho\big([A^*,f]fA+fA^*[f,A]\big)\mathbf{1}_\rho
\end{aligned}$$

So indeed $A_\rho^*A_\rho$ positive close to origin

Then one can conclude... but a bit tedious □

Proof by spectral flow

Use Phillips' result for phase $U = A|A|^{-1}$ and properties of SF:

$$
\begin{aligned}
\mathrm{Ind}(\Pi A \Pi + \mathbf{1} - \Pi) \;&=\; \mathrm{SF}(U^* D U, D) \\
&=\; \mathrm{SF}(\kappa\, U^* D U, \kappa\, D) \\
&=\; \mathrm{SF}\left(\begin{pmatrix} U & 0 \\ 0 & \mathbf{1} \end{pmatrix}^* \begin{pmatrix} \kappa\, D & 0 \\ 0 & -\kappa\, D \end{pmatrix} \begin{pmatrix} U & 0 \\ 0 & \mathbf{1} \end{pmatrix}, \begin{pmatrix} \kappa\, D & 0 \\ 0 & -\kappa\, D \end{pmatrix} \right) \\
&=\; \mathrm{SF}\left(\begin{pmatrix} U & 0 \\ 0 & \mathbf{1} \end{pmatrix}^* \begin{pmatrix} \kappa\, D & \mathbf{1} \\ \mathbf{1} & -\kappa\, D \end{pmatrix} \begin{pmatrix} U & 0 \\ 0 & \mathbf{1} \end{pmatrix}, \begin{pmatrix} \kappa\, D & 0 \\ 0 & -\kappa\, D \end{pmatrix} \right) \\
&=\; \mathrm{SF}\left(\begin{pmatrix} \kappa\, U^* D U & U \\ U^* & -\kappa\, D \end{pmatrix}, \begin{pmatrix} \kappa\, D & 0 \\ 0 & -\kappa\, D \end{pmatrix} \right) \\
&=\; \mathrm{SF}\left(\begin{pmatrix} \kappa\, D & U \\ U^* & -\kappa\, D \end{pmatrix}, \begin{pmatrix} \kappa\, D & 0 \\ 0 & -\kappa\, D \end{pmatrix} \right)
\end{aligned}
$$

Now localize and use $\mathrm{SF} = \frac{1}{2}\,\mathrm{Sig}$ on paths of selfadjoint matrices $\square$

Even pairings (in even dimension)

Consider gapped Hamiltonian H on $\mathcal{H}$ specifying $P = \chi(H \leq 0)$

Dirac operator D on $\mathcal{H} \oplus \mathcal{H}$ is odd w.r.t. grading $\Gamma = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}$

Thus $D = -\Gamma D \Gamma = \begin{pmatrix} 0 & D' \\ (D')^* & 0 \end{pmatrix}$ and Dirac phase $F = D'|D'|^{-1}$

Fredholm operator $PFP + (\mathbf{1} - P)$ has index $=$ Chern number

Spectral localizer

$$L_\kappa = \begin{pmatrix} H & \kappa D' \\ \kappa (D')^* & -H \end{pmatrix} = H \otimes \Gamma + \kappa D$$

Theorem (with Loring 2018)

Suppose $\|[H, D']\| < \infty$ *and* D' *normal, and* κ, ρ *with* (*) *and* (**)

$$\mathrm{Ind}\big(PFP + (\mathbf{1} - P)\big) = \tfrac{1}{2}\,\mathrm{Sig}(L_{\kappa,\rho})$$

Elements of proof

Definition

A fuzzy sphere (X_1, X_2, X_3) *of width* $\delta < 1$ *in* C^**-algebra* $\mathcal{K}$ *is a collection of three self-adjoints in* $\mathcal{K}^+$ *with spectrum in* $[-1, 1]$ *and*

$$\left\| \mathbf{1} - (X_1^2 + X_2^2 + X_3^2) \right\| \;<\; \delta \qquad\qquad \|[X_j, X_i]\| \;<\; \delta$$

Proposition

If $\delta \leq \frac{1}{4}$*, one gets class* $[L]_0 \in K_0(\mathcal{K})$ *by self-adjoint invertible*

$$L \;=\; \sum_{j=1,2,3} X_j \otimes \sigma_j \;\in\; M_2(\mathcal{K}^+)$$

Reason: L invertible and thus has positive spectral projection

Remark: odd-dimensional spheres give elements in $K_1(\mathcal{K})$

Proposition

$L_{\kappa,\rho}$ homotopic to $L = \sum_{j=1,2,3} X_j \otimes \sigma_j$ in invertibles

Construction of that particular fuzzy sphere:

Smooth tapering $f_\rho : \mathbb{R} \to [0,1]$ with $\mathrm{supp}(f_\rho) \subset [-\rho, \rho]$ as above

Define $F_\rho : \mathbb{R} \to [0,1]$ by

$$F_\rho(x)^4 \;+\; f_\rho(x)^4 \;=\; 1$$

If $D' = D_1 + iD_2$ with $D_j^* = D_j$, and $R = |D|$, set

$$\begin{aligned} X_1 &= F_\rho(R)\, R^{-\frac{1}{2}}\, D_{1,\rho}\, R^{-\frac{1}{2}}\, F_\rho(R) \\ X_2 &= F_\rho(R)\, R^{-\frac{1}{2}}\, D_{2,\rho}\, R^{-\frac{1}{2}}\, F_\rho(R) \\ X_3 &= f_\rho(R)\, H_\rho\, f_\rho(R) \end{aligned}$$

Theorem

$$\mathrm{Ind}\,[\pi(P\,F\,P + \mathbf{1} - P)]_1 \;=\; [L_{\kappa,\rho}]_0$$

Proof:

General tool:

Image of K-theoretic index map can be written as fuzzy sphere

$$\mathrm{Ind}[\pi(A)]_1 \;=\; \Big[\sum_{j=1,2,3} Y_j \otimes \sigma_j\Big]_0$$

(by choosing an almost unitary lift A)

Formulas for Y_1, Y_2, Y_3 are explicit (but long)

General tool for $P\,F\,P + \mathbf{1} - P$ provides fuzzy sphere (Y_1, Y_2, Y_3)

Final step: find classical degree 1 map $M : \mathbb{S}^2 \to \mathbb{S}^2$ such that

$$M(Y_1, Y_2, Y_3) \;\sim\; (X_1, X_2, X_3)$$

Numerics for toy model: $p + ip$ superconductor

Hamiltonian on $\ell^2(\mathbb{Z}^2, \mathbb{C}^2)$ depending on μ and δ

$$H = \begin{pmatrix} S_1 + S_1^* + S_2 + S_2^* - \mu & \delta\big(S_1 - S_1^* + \imath(S_2 - S_2^*)\big) \\ \delta\big(S_1 - S_1^* + \imath(S_2 - S_2^*)\big)^* & -(S_1 + S_1^* + S_2 + S_2^* - \mu) \end{pmatrix} + \lambda\, V_{\rm dis}$$

and disorder strength λ and i.i.d. uniformly distributed entries in

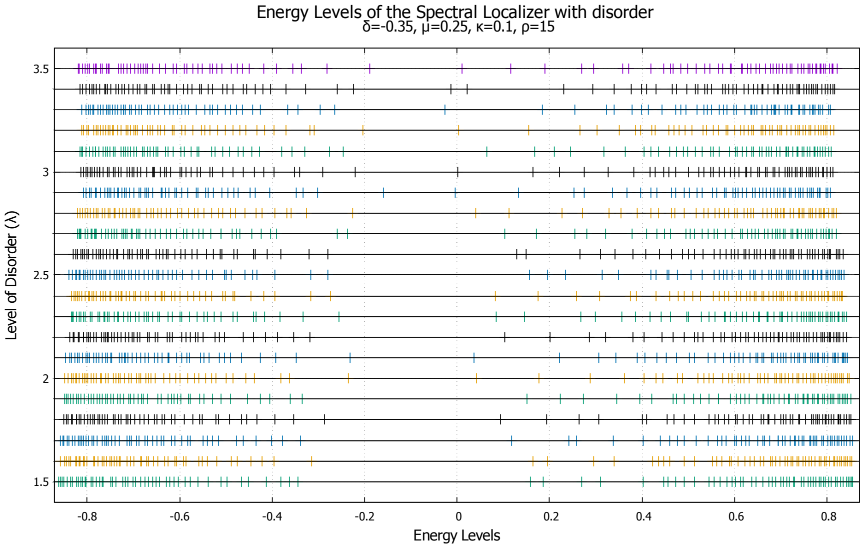
$$V_{\rm dis} = \sum_{n\in\mathbb{Z}^2} \begin{pmatrix} v_{n,0} & 0 \\ 0 & v_{n,1} \end{pmatrix} |n\rangle\langle n|$$

Build even spectral localizer from $D = X_1\sigma_1 + X_2\sigma_2 = -\sigma_3 D \sigma_3$:

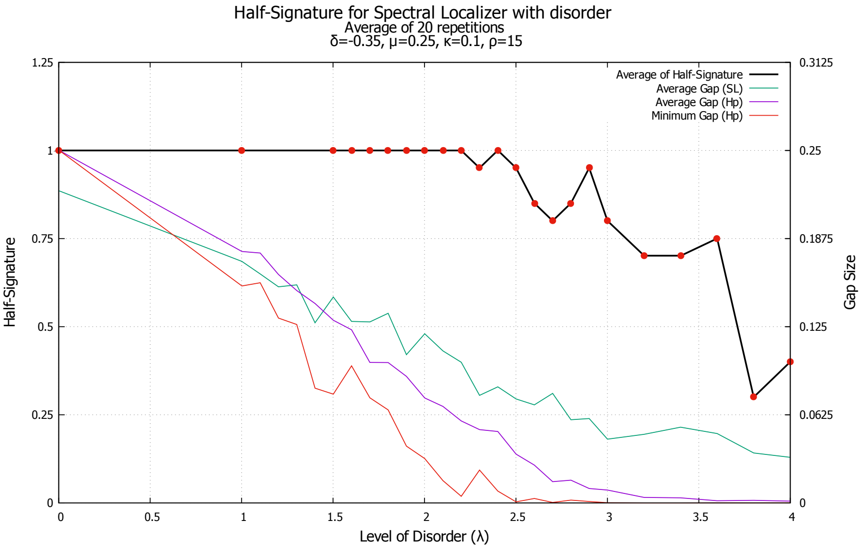
$$L_{\kappa,\rho} = \begin{pmatrix} H_\rho & \kappa\,(X_1 + iX_2)_\rho \\ \kappa\,(X_1 - iX_2)_\rho & -H_\rho \end{pmatrix}$$

Calculation of signature by block Chualesky algorithm

Low-lying spectrum of spectral localizer



Half-signature and gaps for $p + ip$ superconductor



Resumé = Plan of the talk

- Winding number as prototype of an odd index pairing
- Construction of associated spectral localizer
- Main result: invariant as half-signature of spectral localizer
- Proof via spectral flow
- Extension to general odd pairings
- Even dimensional case
- Proof via fuzzy spheres
- Numerical results
- **Implementation of symmetries**: e.g. $\mathrm{sgn}(\mathrm{Pf}(L_{\kappa,\rho})) \in \mathbb{Z}_2$

Implementation of real symmetries

If A, H have real symmetry (like PHS or TRS), often $\mathrm{Sig}(L_{\kappa,\rho}) = 0$

But

$$\mathrm{sgn}(\det(L_{\kappa,\rho})) \in \mathbb{Z}_2 \qquad , \qquad \mathrm{sgn}(\mathrm{Pf}(L_{\kappa,\rho})) \in \mathbb{Z}_2$$

Not general case (paper) but example: Class CII has odd PHS

$$S^* \, \overline{A} \, S \;=\; A \qquad , \qquad \overline{S} \;=\; S \,, \quad S^2 \;=\; -\mathbf{1}$$

where overline is a real structure on complex Hilbert space

and in $d = 3$ Dirac $D = X_1\sigma_1 + X_2\sigma_2 + X_3\sigma_3$ has odd PHS

$$\Sigma^* \, \overline{D} \, \Sigma \;=\; -\, D \qquad , \qquad \Sigma \;=\; i\sigma_2$$

Hence with $R = \Sigma \otimes S$

$$R^* \, \overline{L_\kappa} \, R \;=\; -\, L_\kappa \qquad , \qquad R^2 \;=\; \mathbf{1}$$

Thus $\mathrm{sgn}(\mathrm{Pf}(L_{\kappa,\rho})) \in \mathbb{Z}_2$

Theorem (General tool)

$0 \to \mathcal{K} \hookrightarrow \mathcal{B} \stackrel{\pi}{\to} \mathcal{Q} \to 0$ *short exact sequence with* $\mathcal{Q}$ *unital*

$A \in \mathcal{B}$ *contraction with* $\pi(A) \in \mathcal{Q}$ *invertible, so* $[\pi(A)]_1 \in K_1(\mathcal{Q})$

Assume $A = A_1 + i\,A_2$ *almost normal, namely* $\|[A_1, A_2]\| < \epsilon$

Choose smooth $\psi : [0,1] \to [0,1]$ *and* $\phi : [0,1] \to [-1,1]$ *such that*

$$\phi(1) \,=\, 1 \,=\, -\phi(0) \qquad , \qquad x^2\,\psi(x)^4 \,+\, \phi(x)^2 \,=\, 1$$

like $\phi(x) = 2x^2 - 1$ *and* $\psi(x) = 2^{\frac{1}{2}}(1 - x^2)^{\frac{1}{4}}$. *Set* $B = (A_1^2 + A_2^2)^{\frac{1}{2}}$,

$$Y_1 \,=\, \psi(B)A_1\psi(B) \quad , \quad Y_2 \,=\, -\,\psi(B)A_2\psi(B) \quad , \quad Y_3 \,=\, \phi(B)$$

Then (Y_1, Y_2, Y_3) *fuzzy sphere in* $\mathcal{K}$ *giving* K*-theoretic index map:*

$$\mathrm{Ind}[\pi(A)]_1 \;=\; \Big[\sum_{j=1,2,3} Y_j \otimes \sigma_j\Big]_0$$