



$M \cap \Phi$

2018

Hecke Relations in  
Rational Conformal Field Theory  
(with Y. Wu, arXiv:1804.06860)

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# INTRODUCTION

This talk concerns a new relation between characters of two-dimensional rational conformal field theory (RCFT) given by Hecke operators.

The relation generalizes previously known Galois symmetry relations between the representations of the modular group provided by RCFT characters.

I will discuss these new relations, explain how they explain a number of scattered results in the literature and present some possible applications.

# Examples:

The Yang-Lee model is a non-unitary minimal model  $M(5, 2)$  (RCFT) with two independent characters

$$\chi_0^{YL} = q^{-1/60} \sum_{n=0} c_0^{YL}(n) q^n$$

$$\chi_{1/5}^{YL} = q^{11/60} \sum_{n=0} c_{1/5}^{YL}(n) q^n$$

Affine  $G_2$  is a unitary rational CFT with two independent characters

$$\chi_0^{G_2} = q^{-7/60} \sum_{n=0} c_0^{G_2}(n) q^n$$

$$\chi_{17/60}^{G_2} = q^{17/60} \sum_{n=0} c_{17/60}^{G_2}(n) q^n$$

Although apparently unrelated, there is a subtle relation between the character coefficients:

# DATA

coefficients of  $q$  expansion of  $h=1/5$  character of Yang-Lee model  $c=-22/5$

coefficients of  $q$  expansion of vacuum character of affine  $G_2$  at level one ( $c=14/5$ ) divided by 7

$n \backslash k$	1	2	3	4	5	6	7
0	0	1	1	1	1	2	2
1	3	3	4	4	6	6	8
2	9	11	12	15	16	20	22
3	26	29	35	38	45	50	58
4	64	75	82	95	105	120	133
5	152	167	190	210	237	261	295
6	324	364	401	448	493	551	604
7	673	739	820	899	997	1091	1207
8	1321	1457	1593	1756	1916	2108	2301
9	2525	2753	3019	3287	3599	3917	4281
10	4655	5084	5521	6021	6537	7118	7721
11	8401	9103	9894	10715	11631	12587	13653
12	14761	15995	17285	18710	20203	21854	23579
13	25483	27480	29671	31975	34502	37153	40058
14	43114	46447	49958	53787	57815	62202	66826
15	71844	77140	82885	88939	95502	102428	109914
16	117820	126362	135376	145105	155382	166450	178148
17	190741	204038	218343	233460	249691	266850	285266
18	304722	325589	347644	371264	396235	422966	451209
19	481424	513361	547483	583553	622077	662780	706228
20	752141	801100	852841	907990	966247	1028311	1093881
21	1163672	1237403	1315853	1398699	1486806	1579853	1678733
22	1783153	1894077	2011173	2135511	2266766	2406046	2553058
23	2709006	2873560	3048045	3232148	3427249	3633082	3851139
24	4081118	4324669	4581514	4853375	5140036	5443362	5763114
25	6101337	6457834	6834755	7231982	7651842	8094202	8561620
26	9054025	9574106	10121903	10700327	11309427	11952388	12629349
27	13343681	14095665	14888948	15723845	16604348	17530906	18507742

Discrepancy

2
6
20
50
120
261
3858/7
1091
2108
3917
7118
12587
21854
260072/7
62202
102428
166450
266850
422966
662780
7198178/7
1579853
2406046
3633082
5443362
8094202
11952388
122716344/7

1/7

1/7

1/7

2/7

Table 4: Coefficients  $c_{1/5}^{YL}(7n+k)$

# RCFT

Hilbert space:

$$\mathcal{H} = \bigoplus_{i, \bar{i}} \mathcal{N}_{i, \bar{i}} V_i \otimes \overline{V_{\bar{i}}}$$

Representation of chiral algebra, e.g. Virasoro for minimal models

Characters:

$$\chi_i(\tau) = \text{Tr}_{V_i} q^{L_0 - c/24}, \quad q = e^{2\pi i \tau}$$

Partition function:

$$Z(\tau) = \sum_{i \in \mathcal{I}, \bar{i} \in \overline{\mathcal{I}}} \mathcal{N}_{i, \bar{i}} \chi_i(\tau) \overline{\chi_{\bar{i}}(\tau)}$$

Examples:

Ising model, Yang-Lee model, affine Lie algebras, Monster VOA, BM VOA

# Modular Properties

Characters are weakly holomorphic weight zero vector-valued modular functions transforming according to

$$\rho : SL(2, \mathbb{Z}) \rightarrow GL(n, \mathbb{C})$$

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} (\tau \rightarrow -1/\tau) \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} (\tau \rightarrow \tau + 1)$$

$$(\rho(S))^2 = (\rho(S)\rho(T))^3 = C \quad C^2 = 1$$

$$N = \text{order}(\rho(T)) \quad \Gamma(N) \subset \ker(\rho) \quad (\text{Bantay})$$

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \mid a = d = 1 \pmod{N}, b = c = 0 \pmod{N} \right\}$$

# Example

## Ising Model

$$\chi_0^I = \frac{1}{2} \left( \sqrt{\frac{\theta_3(\tau)}{\eta(\tau)}} + \sqrt{\frac{\theta_4(\tau)}{\eta(\tau)}} \right) = q^{-1/48} (1 + q^2 + q^3 + 2q^4 + 2q^5 + \dots),$$

$$\chi_{1/2}^I = \frac{1}{2} \left( \sqrt{\frac{\theta_3(\tau)}{\eta(\tau)}} - \sqrt{\frac{\theta_4(\tau)}{\eta(\tau)}} \right) = q^{23/48} (1 + q + q^2 + q^3 + 2q^4 + 2q^5 + \dots),$$

$$\chi_{1/16}^I = \frac{1}{\sqrt{2}} \sqrt{\frac{\theta_2(\tau)}{\eta(\tau)}} = q^{1/24} (1 + q + q^2 + 2q^3 + 2q^4 + 3q^5 + \dots).$$

$$\rho(S) = \begin{pmatrix} 1/2 & 1/2 & 1/\sqrt{2} \\ 1/2 & 1/2 & -1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{pmatrix} \quad \chi_i(-1/\tau) = \sum_j \rho(S)_{ij} \chi_j(\tau)$$


$$\rho(T) = e^{-2\pi i/48} \text{diag}(1, e^{2\pi i/2}, e^{2\pi i/16}) \quad \chi_i(\tau + 1) = \sum_j \rho(T)_{ij} \chi_j(\tau)$$

$$N = 48$$

# Galois Symmetry

$K$ =Field extension of  $\mathbb{Q}$  by matrix elements of  $\rho(\gamma)$

De Boer & Goeree plus Kronecker-Weber:

  $K \subseteq \mathbb{Q}[\zeta_N]$      $N$ =conductor (also of RCFT).

$$\text{Gal}(\mathbb{Q}[\zeta_N]) \cong (\mathbb{Z}/N\mathbb{Z})^\times$$

Coste-Gannon: For each

$$\ell \in (\mathbb{Z}/N\mathbb{Z})^\times$$

$$f_{N,\ell} : \rho(T) \rightarrow \rho(T)^\ell$$

$$\rho(S) \rightarrow G_\ell \rho(S)$$

Primitive  $N$ th  
root of unity

Group of units  
in Ring  $\mathbb{Z}/N\mathbb{Z}$

$$f_{N,\ell} : \rho(S)_{i,j} \rightarrow \varepsilon_\ell(i) \rho(S)_{\pi_\ell(i),j} = \varepsilon_\ell(j) \rho(S)_{i,\pi_\ell(j)}$$

 permutation of indices     $\varepsilon = \pm 1$

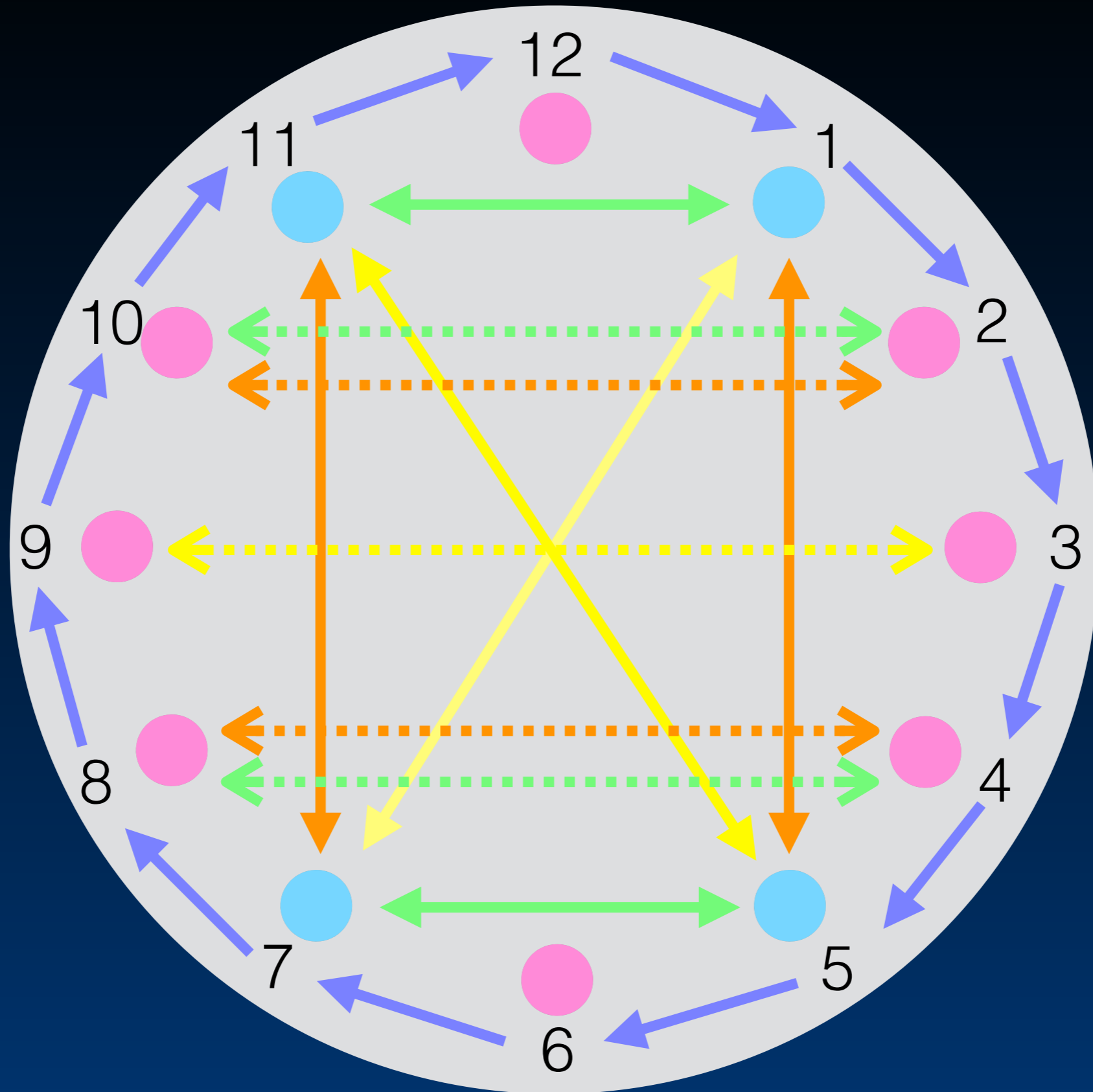


$\mathbb{Z}/12\mathbb{Z}$

$(\mathbb{Z}/12\mathbb{Z})^\times$

5 7 11

$f_5$   $f_7$   $f_{11}$



# Examples of Galois RCFT Relations

$$\rho(S)^{YL} = \begin{pmatrix} -\frac{1}{2 \sin(\pi/5)} & \frac{1}{2 \sin(2\pi/5)} \\ \frac{1}{2 \sin(2\pi/5)} & \frac{1}{2 \sin(\pi/5)} \end{pmatrix} \quad \rho(S)^{G_2} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \rho(S)^{YL}$$



$$\rho(T)^{YL} = \text{diag}(e^{2\pi i 11/60}, e^{-2\pi i/60}) \quad \rho(T)^{G_2} = (\rho(T)^{YL})^7$$

Similarly Yang-Lee



Three character RCFT    Ising  $\xrightarrow{f_{48,47}}$  Baby Monster

These are relations between modular representations.  
We will extend them to relations between **characters**.

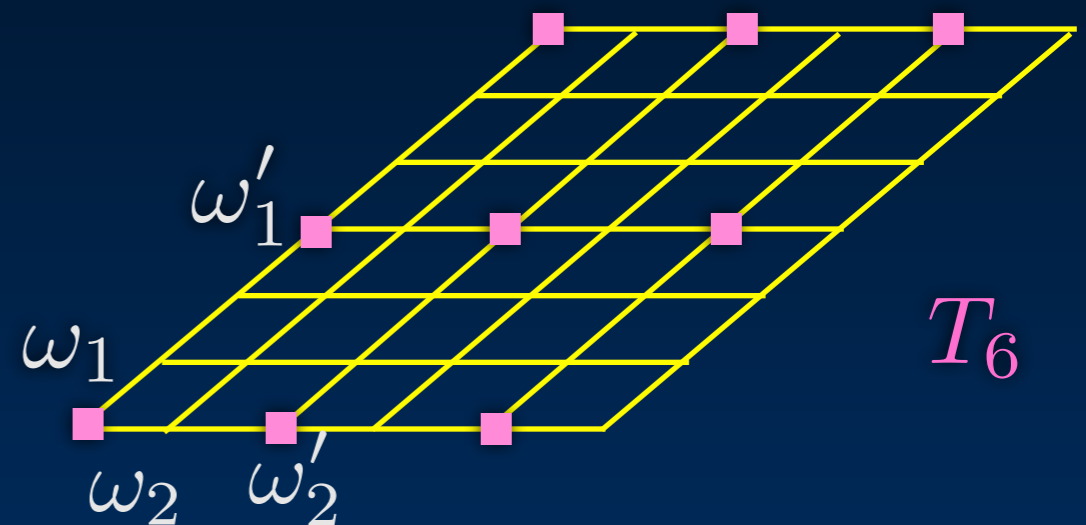
# Scalar Hecke

Modular forms can be thought of in two ways:

Functions of  $\tau \in \mathbb{H}$        $f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau)$

Functions of rank 2 lattices       $F(\lambda L) = \lambda^{-k} F(L)$

$$F(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2) = \omega_2^{-k} f(\omega_1/\omega_2)$$



Hecke operator:  $(T_n F)(L) = \sum_{\substack{L' \subset L \\ |L/L'|=n}} F(L')$

$$\omega'_1 = a\omega_1 + b\omega_2$$

$$\omega'_2 = c\omega_1 + d\omega_2$$

$$ad - bc = n$$

# Hecke Operators

When translated into an action on modular forms as a function of  $\tau$  or  $q = e^{2\pi i\tau}$  this leads to

$$f = \sum_n a(n)q^n \quad (T_p f)(\tau) = \sum_n a^{(p)}(n)q^n$$

Action on  
Fourier  
coefficients:

$$a^{(p)}(n) = \begin{cases} p^k a(pn) & \text{if } p \nmid n, \\ p^{k-1} (p a(pn) + a(n/p)) & \text{if } p|n. \end{cases}$$

To generalize Hecke operators to RCFT characters:

Use Hecke operators for  $\Gamma(N)$  (Rankin, Chap. 9)

Use modular representation properties of RCFT characters.

This leads to the following formula for Hecke images of RCFT characters for  $(p, N) = 1$ :

On Fourier coefficients:

$$\chi_i(\tau) = \sum_n b_i(n) q^{n/N}$$
$$(\mathbb{T}_p \chi)_i(\tau) = \sum_n b_i^{(p)}(n) q^{n/N}$$

$$b_i^{(p)}(n) = \begin{cases} pb_i(np) & p \nmid n \\ pb_i(np) + \sum_b \rho_{ij}(\sigma_p) b_j(n/p) & p | n \end{cases}$$

The  $(\mathbb{T}_p \chi)_i$  are again modular forms for  $\Gamma(N)$  but transform under a different representation of  $SL(2, \mathbb{Z})$

where  $\sigma_p$  is the pre-image of  $\begin{pmatrix} \bar{p} & 0 \\ 0 & p \end{pmatrix}$  under the mod  $N$  map  $SL(2, \mathbb{Z}) \rightarrow SL(2, \mathbb{Z}/N\mathbb{Z})$  and  $\bar{p}p = 1 \pmod{N}$

Example:  $N = 60, p = 7, \bar{p} = 43$        $\sigma_7 = \begin{pmatrix} 27343 & -33780 \\ 480 & -593 \end{pmatrix}$

Representation  
of Hecke image

$$\rho^{(p)}(S) = \rho(\sigma_p S), \quad \rho^{(p)}(T) = \rho(T^{\bar{p}})$$

The change of representation under Hecke is the same as that under Galois for  $\ell = p, (p, N) = 1$ .

The equivalence relies on the identities

$$\begin{aligned} f_{N,p}(\rho(S)) &= \rho(\sigma_{\bar{p}} S) \\ f_{N,p}(\rho(T)) &= \rho(T^p) \end{aligned}$$

# Applications and Examples

The example from the first DATA slide:

$$\chi_0^{YL} = q^{-1/60} G(q) = q^{-1/60} \sum_{n=0}^{\infty} c_0^{YL}(n) q^n$$

$$\chi_{1/5}^{YL} = q^{11/60} H(q) = q^{11/60} \sum_{n=0}^{\infty} c_{1/5}^{YL}(n) q^n$$

Then we have the Hecke relation  $\chi^{G_2} = T_7 \chi^{YL}$

$$c_0^{G_2}(n) = \begin{cases} 7c_{1/5}^{YL}(7n-1) & \text{if } 7 \nmid n, \\ 7c_{1/5}^{YL}(7n-1) + c_0^{YL}(\frac{n}{7}) & \text{if } 7|n; \end{cases} \quad \rho^{YL}(\sigma_7) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$


$$c_{2/5}^{G_2}(n) = \begin{cases} 7c_0^{YL}(7n+2) & \text{if } 7 \nmid (n-1), \\ 7c_0^{YL}(7n+2) - c_{1/5}^{YL}(\frac{n-1}{7}) & \text{if } 7|(n-1). \end{cases}$$

# Modular Linear Differential Equation

MLDEs are a useful tool for studying and classifying characters of RCFT (Anderson-Moore, Eguchi-Ooguri, Mathur, Mukhi, Sen; Kaneko-Zagier,...)

Ramanujan-Serre:  $\mathcal{D}_k = d/d\tau - \frac{1}{6}i\pi k E_2$ :  $M_k(\Gamma) \rightarrow M_{k+2}(\Gamma)$

quasi modular weight 2  
Eisenstein series



Second order:  $D^2 f - \frac{E_2}{6} Df - \frac{\mu E_4}{4} f = 0$        $D = \frac{1}{2\pi i} \frac{d}{d\tau}$

Solutions ( $\ell = 0$ ) completely classified by Mathur, Mukhi, Sen

$$X = \{YL, A_1, A_2, G_2, D_4, F_4, E_6, E_7, E_{7\frac{1}{2}}\}$$


Yang-Lee  
model



Affine level 1 characters  
Deligne exceptional series



Characters of  
Intermediate Vertex  
Subalgebra  
(Kawasesu)





# More Examples

Two character theories:

$$\chi^{F_4} = T_{13}\chi^{YL} \quad \chi^{E_7} = T_7\chi^{A_1} \quad \chi^{E_7\frac{1}{2}} = T_{19}\chi^{YL}$$

Three character theories: No classification I am aware of. Examples include

Minimal models:  $\mathcal{M}_{4,3}$  (Ising),  $\mathcal{M}_{5,2}^{\otimes 2}$  ( $YL^{\otimes 2}$ ),  $\mathcal{M}_{7,2}$

Models w/o Kac-Moody symmetry (Hampapura-Mukhi):

BabyMonster	$c = 47/2 = 47 \times 1/2$	“Duality” of H-M and relations implied by Hecke relations
	$c = 164/5 = 41 \times 4/5$	
	$c = 236/7 = 59 \times 4/7$	

One can also use Hecke images to construct families of possible RCFT characters. As an example, if

$$p = 7, 13, 47, 53 \pmod{60}$$

then  $T_p \chi^{YL}$  obey

1. Have non-negative integer coefficients in  $q$  expansion.
2. The vacuum appears with degeneracy one.
3. The fusion coefficients from Verlinde are non-negative.

Consistency with Virasoro requires

$$\chi_{N,p} = T_{60*N+p} \chi^{YL} + \sum_{k=0}^{N-2} d(k) T_{p+60k} \chi^{YL}$$

# SUMMARY

Many RCFT with the same  $n$ =dimension of modular rep,  $N$ =order of  $\rho(T)$  and with effective central charges  $c_{eff}^{(1)} = pc_{eff}^{(2)}$  have modular representations and characters related by Hecke operators  $T_p$ ,  $(p, N) = 1$ . This generalizes and extends Galois symmetry.

We have found and studied such relations for  $n=2,3$  involving minimal models, affine Lie algebras, Gaussian models and their orbifolds and certain moonshine CFTs. These relations appear to be very common.

# Questions

Do these Hecke operators have a natural physical origin?

Can one understand more systematically which RCFT with the same  $n, N$  have characters related by Hecke operators?

Does this indicate there are new symmetries acting on the space of RCFTs?

RCFTs provide building block for construction of SCFTs describing special points in CY moduli space. Are there new geometrical relations following from Hecke relations?

RCFTs and their fusion algebras, modular tensor categories, characters etc. appear in many places in condensed matter physics:

Boundary modes of QHE systems and topological insulators.

Tool for computing and studying Entanglement Entropy.

Quantum computation.

It will be interesting to see if these new Hecke relations have implications in the real world.

**THANK YOU**

# Extra Slides

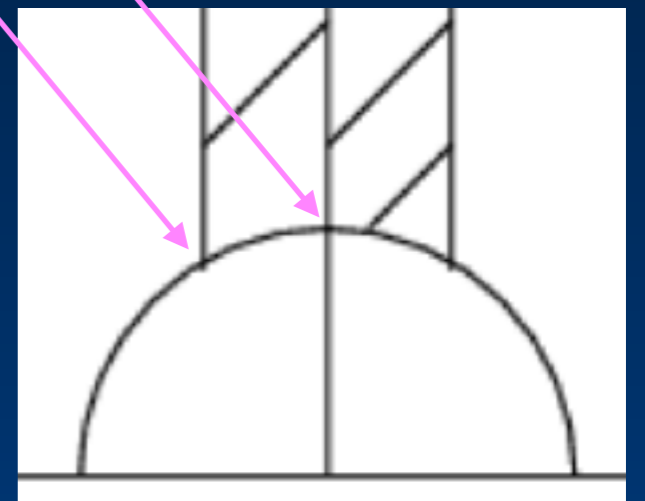
# Hecke and MLDE

$$\mathcal{D}^n f + \sum_{k=0}^{n-1} \phi_k(\tau) \mathcal{D}^k f = 0 \quad \phi_k = (-1)^k W_k / W$$

$$W_k = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ \mathcal{D}f_1 & \mathcal{D}f_2 & \cdots & \mathcal{D}f_n \\ \vdots & \vdots & & \vdots \\ \mathcal{D}^{k-1}f_1 & \mathcal{D}^{k-1}f_2 & \cdots & \mathcal{D}^{k-1}f_n \\ \mathcal{D}^{k+1}f_1 & \mathcal{D}^{k+1}f_2 & \cdots & \mathcal{D}^{k+1}f_n \\ \vdots & \vdots & & \vdots \\ \mathcal{D}^n f_1 & \mathcal{D}^n f_2 & \cdots & \mathcal{D}^n f_n \end{vmatrix}, \quad W = W_n$$

$$\text{ord}_\infty(W) + \frac{\ell(W)}{6} = \frac{k}{12}$$

$\omega$       $i$



$$\ell(W) = 6 \left( \frac{1}{2} \text{ord}_i(W) + \frac{1}{3} \text{ord}_\omega(W) + \sum'_{p \in \mathcal{F}} \text{ord}_p(W) \right)$$



Mathur, Mukhi & Sen classified  $n=2$ ,  $\ell(W) = 0$  solutions, but solutions exist for  $\ell(W) > 0$  and are Hecke images of  $\ell(W) = 0$  solutions since  $T_p$  changes  $\text{ord}_\infty(W)$

$p$	7	11	13	17	19	23	29	31	37	41	43	47	53	59	61	67	71
$\ell^{YL}(p)$	0	2	0	2	0	2	2	0	0	2	0	2	2	2	6	6	8

Table 1: Number of zeros in the modular Wronskian for Hecke images under  $T_p$  of Yang-Lee characters for small values of  $p$ .

- These Hecke images have negative coefficients
- These Hecke images have positive coefficients and as RCFT characters appear in work of Naculich and Hampapurma & Mukhi

$p$	5	7	11	13	17	19	23	25	29	31	35	37	41	43	47	49	53	57
$\ell^I(p)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	6	6	6

Table 1: Number of zeros in the modular Wronskian for Hecke images under  $T_p$  of Ising characters for small values of  $p$ .

In math these are often applied to weight  $k > 0$  modular forms and used to construct cusp forms which are eigenfunctions of the Hecke operators. E.g. the unique weight 12 cusp form (i.e. vanishing as  $\tau \rightarrow i\infty$ )

$$\Delta = \eta^{24} = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum_{n=1}^{\infty} c(n) q^n$$

$$T_n \Delta = c(n) \Delta$$

The modularity theorem used in the proof of Fermat's last theorem associates a weight 2 Hecke eigenform to each elliptic curve over the rationals.

In RCFT we are interested in the action on weight 0, weakly holomorphic modular functions which are never Hecke eigenfunctions.

