Duality for integrable systems associated to quantum toroidal algebras

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The transfer matrices

Let U_q be your favorite quantum group.
Let $\mathcal{R} \in U_q \,\tilde{\otimes}\, U_q$ be the R-matrix satisfying the Yang-Baxter equation

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}.$$

Let $Q \in U_q$ be the twist operator:

$$\mathcal{R}(Q \otimes Q) = (Q \otimes Q)\mathcal{R}.$$

Let V be an *admissible* U_q-module. Then the trace

$$T_V = (\mathrm{Tr_V} \otimes 1)\,((Q \otimes 1)\mathcal{R}) \in \tilde{U}_q$$

is called the transfer matrix.

Lemma. *For any admissible modules V_1, V_2, the transfer matrices commute:* $T_{V_1}T_{V_2} = T_{V_2}T_{V_1}$.

The XXZ type models

$$\text{Recall}: \quad T_V = (\mathrm{Tr_V} \otimes 1)\,((Q \otimes 1)\mathcal{R}).$$

Thus, the R matrix gives an *embedding* of the Grothendick ring of admissible representations to the quantum group:

$$T: \ K_0(Rep\, U_q) \to \tilde{U}_q, \qquad V \ \mapsto T_V.$$

The image $\mathcal{B}_q = \mathrm{Im}(T)$ is the commutative algebra of quantum Hamiltonians.

The algebra $\mathcal{B}_q$ acts on an appropriate class of representations of U_q.

Problem. (XXZ type models) Understand the spectrum of $\mathcal{B}_q$.

The Gaudin type models

The limit $q \to 1$ gives an algebra of quantum Hamiltonians in the corresponding universal enveloping algebra:

$$\mathcal{B} = \lim_{q\to 1} \mathcal{B}_q \in \tilde{U}.$$

The limit is not easy. There are alternative constructions (of the same algebra) for affine Lie algebras:

- from the center on the critical level [FFR];
- from Segal-Sugawara vectors in the vacuum modules [M];
- shift of argument method [R].

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B.Feigin, E.Frenkel, and
N. Reshetikhin, (94)

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An example

Let $U = \mathfrak{gl}_n[t] = \mathfrak{gl}_n \otimes \mathbb{C}[t]$.

We use formal series $e_{ij}(x) = \sum_{s=0}^{\infty} (e_{ij} \otimes t^s) x^{-s-1} \in U[[x^{-1}]]$.

Let $\bar{Q} = \sum_{i=1}^{n} u_i e_{ii}$.

Consider the matrix

$$E_n^u = \begin{pmatrix} \partial_x - u_1 - e_{11}(x) & -e_{21}(x) & \dots & -e_{n1}(x) \\ -e_{12}(x) & \partial_x - u_2 - e_{22}(x) & \dots & -e_{n2}(x) \\ \dots & \dots & \dots & \dots \\ -e_{1n}(x) & -e_{2n}(x) & \dots & \partial_x - u_n - e_{nn}(x) \end{pmatrix}.$$

Expand the row determinant:

$$\operatorname{rdet} E_n^u = \partial_x^n + B_1(x)\partial_x^{n-1} + B_2(x)\partial_x^{n-2} + \dots + B_n(x).$$

Theorem.([T]) *Coefficients of $B_i(x)$ commute and generate the algebra $\mathcal{B}_n^u$ of quantum Hamiltonians in $\mathfrak{gl}_n[t]$.*

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The classical $\mathfrak{gl}_m - \mathfrak{gl}_n$ duality

Consider the vector space $V = \mathbb{C}[x_{ij}]_{i=1,\dots,m}^{j=1,\dots,n}$.

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$$\begin{tikzcd}
\mathfrak{gl}_m \arrow[r, bend left] & \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \dots & \dots & \dots & \dots \\ x_{m1} & x_{m2} & \dots & x_{mn} \end{pmatrix} & \mathfrak{gl}_n \arrow[l, bend right]
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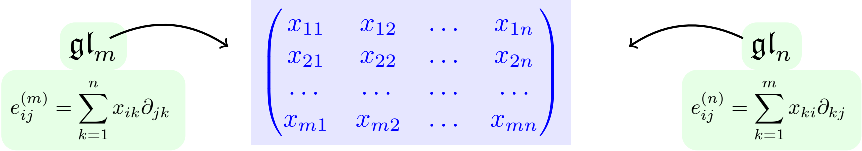

$\mathfrak{gl}_m$

$$e_{ij}^{(m)} = \sum_{k=1}^{n} x_{ik}\partial_{jk}$$

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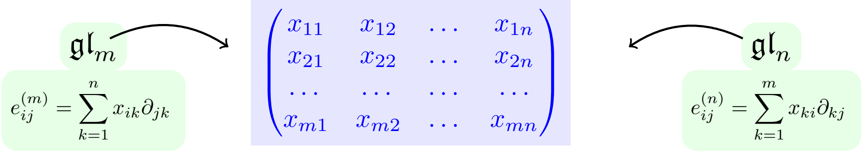

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Lemma. *As a $\mathfrak{gl}_m$ module,* $V = \displaystyle\bigoplus_{k_1,\dots,k_n=0}^{\infty} L_{k_1\omega_1}^{(m)} \otimes \dots \otimes L_{k_n\omega_1}^{(m)}$.

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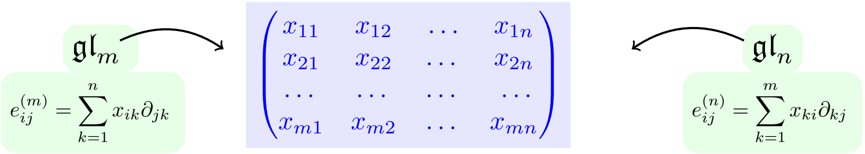
Lemma. *We have* $[\mathfrak{gl}_m, \mathfrak{gl}_n] = 0$ *in* $\mathrm{End}(V)$.

The $\mathfrak{gl}_n - \mathfrak{gl}_m$ duality of Gaudin models

Choose complex evaluation parameters.



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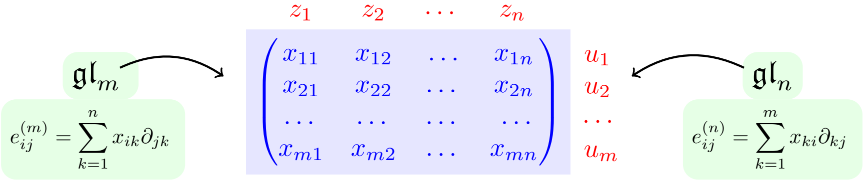

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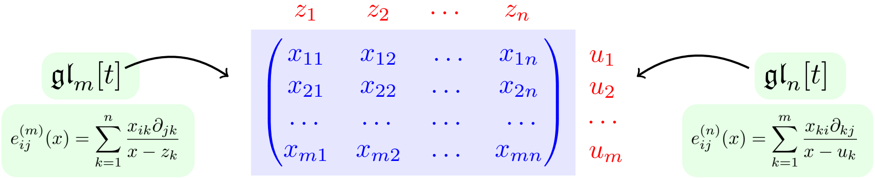

$\mathfrak{gl}_m[t]$

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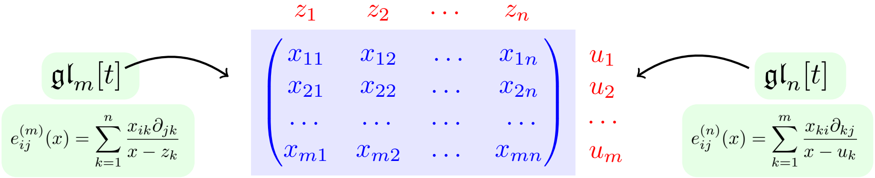

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Theorem.([MTV]) *The algebras of quantum Hamiltonians in* $\mathrm{End}(V)$ coincide: $\mathcal{B}_m^u = \mathcal{B}_n^z$.

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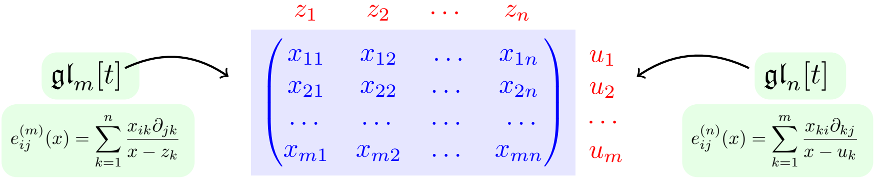

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Theorem.

E.M., V. Tarasov, and
A. Varchenko, (06)

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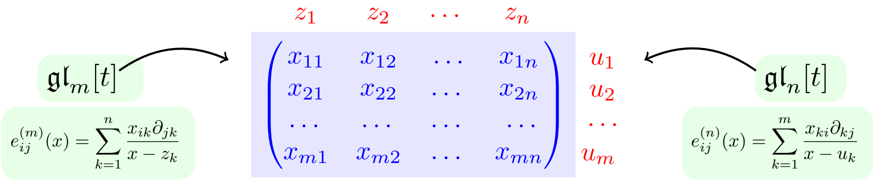
The correspondence of quantum Hamiltonians



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Corollary. Eigenvectors of $\mathcal{B}_m^u$ and of $\mathcal{B}_n^z$ coincide.

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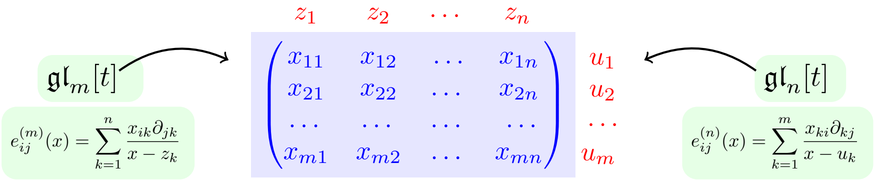

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Write: $\prod_{i=1}^{n}(x-z_i)\,\mathrm{rdet}\,E^u_m=\sum_{i=1}^{n}\sum_{j=1}^{m}A^{(m)}_{ij}x^i\partial^j$, where $A^{(m)}_{ij}\in\mathrm{End}(V)$.

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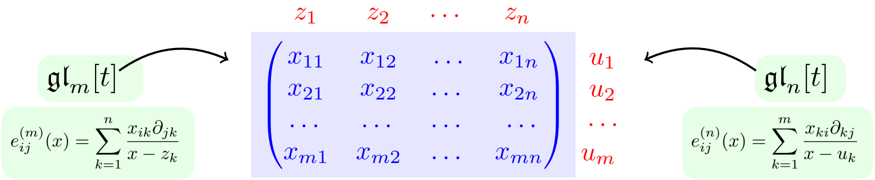

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Corollary. Eigenvectors of $\mathcal{B}_m^u$ and of $\mathcal{B}_n^z$ coincide.

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Theorem. ([MTV]) *We have* $A_{ij}^{(m)}=A_{ji}^{(n)}$.

The correspondence of quantum Hamiltonians

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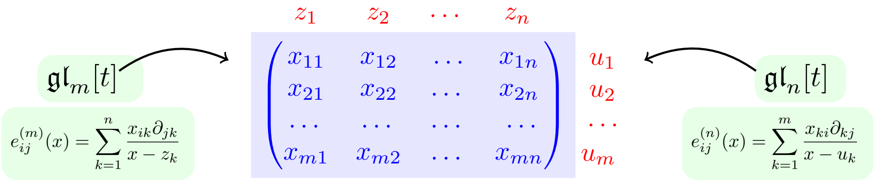
E.M., V. Tarasov, and A. Varchenko, (05)

The correspondence of quantum Hamiltonians



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Quantum toroidal algebras

Let $\mathcal{E}_m(q_1,q)$ be the quantum toroidal algebra associated to $\mathfrak{gl}_m$, [GKV].

- The algebra $\mathcal{E}_m(q_1,q)$ is an affinization of $U_q\widehat{\mathfrak{gl}}_m$.
- The algebra $\mathcal{E}_m(q_1,q)$ has generators $E_i(z), F_i(z), K_i^{\pm}(z)$, $i=0,\dots,m-1$, central element q^c and degree operator q^d.
- For any j, $E_i(z), F_i(z), K_i^{\pm}(z)$ $(i\neq j)$, $K_j^{\pm}(z)$, q^c, q^d, generate a subalgebra canonically isomorphic to $U_q\widehat{\mathfrak{gl}}_m$ in Drinfeld new realization. The one for $j=0$ is called the vertical subalgebra.
- The zero modes $E_{i,0}, F_{i,0}, K_{i,0}^{\pm}$ generate a subalgebra canonically isomorphic to level zero $U_q\widehat{\mathfrak{gl}}_m$ in Drinfeld-Jimbo realization. It is called the horizontal subalgebra.

Introduce the twist operator $Q=p_0^d\prod_{i=1}^{m-1}p_i^{-\Lambda_i}$.

Let $\widehat{B}_m^p$ be the corresponding algebra of quantum Hamiltonians.

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V. Ginzburg, M. Kapranov, and E. Vasserot, (95)

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The Fock module

The quantum toroidal algebra $\mathcal{E}_m(q_1, q)$ has a family of Fock representations $\mathcal{F}_i(z, t, k)$.

- The Fock module restricted to vertical $U_q\widehat{\mathfrak{gl}}_m$ is the integrable module of level $c = 1$ with highest weight Λ_i.
- The degree of the highest vector is t.
- The central element $q^{\sum_{i=0}^{m-1} \epsilon_i}$ acts by q^k.
- The Fock module has a realization by vertex operators, [S].
- The Fock module has a realization by Macdonald type operators, [FJMM].

The quantum Hamiltonian corresponding to module $\mathcal{F}_i(z, t, k)$ is computed explicitly. The coefficient I_s of z^s is given by an mk-fold integral.

Example. For $m = 1$, $I_1 = \displaystyle\int_{|x|=1} F(x) \prod_{s=0}^{\infty} \bar{K}^+(p^{-s}x)dx/x$

(in the region $|q_1| < 1 < |q_1 q^2|$ and by analytic continuation everywhere else).

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Y. Saito, (98)

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B. Feigin, M. Jimbo, T. Miwa, and E.M. (12)

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The $U_q\widehat{\mathfrak{gl}}_m - U_q\widehat{\mathfrak{gl}}_n$ duality

Let $H_{ij}(x)$ be free bosons: $H_{ij}(x)H_{kl}(y) \sim \delta_{ik}\delta_{jl}/(x-y)^2$.
Consider the vector space $V = \mathbb{C}[H^+_{ij}(x)]^{j=1,\dots,n}_{i=1,\dots,m} \otimes \mathbb{C}(\mathbb{Z}^{mn})$.

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Then one can define [JF], [FJM]:

$$\begin{tikzcd}
U_q\widehat{\mathfrak{gl}}_m \arrow[r, bend left] & \begin{pmatrix} H_{11}^{+}(x) & H_{12}^{+}(x) & \dots & H_{1n}^{+}(x) \\ H_{21}^{+}(x) & H_{22}^{+}(x) & \dots & H_{2n}^{+}(x) \\ \dots & \dots & \dots & \dots \\ H_{m1}^{+}(x) & H_{m2}^{+}(x) & \dots & H_{mn}^{+}(x) \end{pmatrix} & U_q\widehat{\mathfrak{gl}}_n \arrow[l, bend right]
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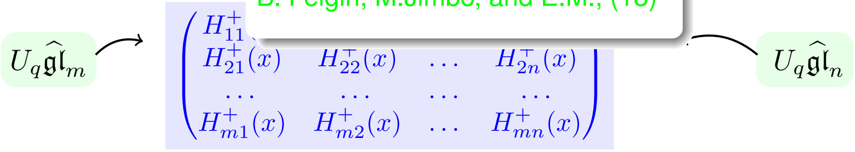
I. Frenkel and N. Jing, (88)

B. Feigin, M.Jimbo, and E.M., (18)



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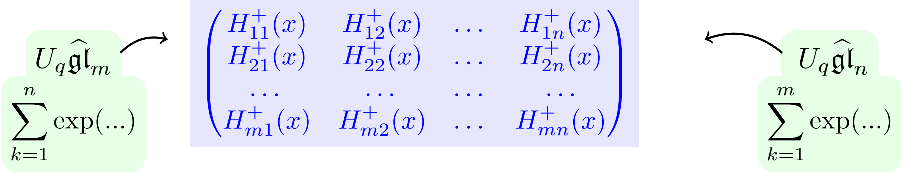

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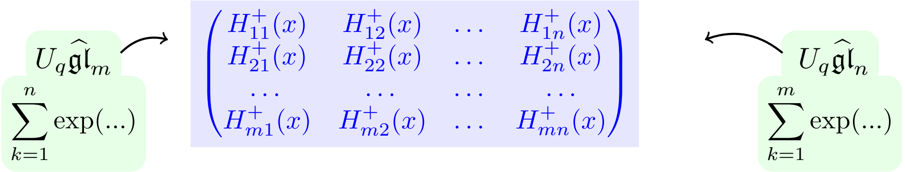

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Lemma. *As a* $U_q\widehat{\mathfrak{gl}}_m$ *module,* $V = \bigoplus_{k_1,\dots,k_n=0}^{\infty} L^{(m)}_{\Lambda_{r(k_1)}}(t(k_1),k_1) \otimes \dots \otimes L^{(m)}_{\Lambda_{r(k_n)}}(t(k_n),k_n)$.

Here $r(k) = \mathrm{res}\ k \pmod m$, $m\,l(k) = k - r(k)$, $2t(k) = r(k)(l(k)+1) + (m-r(k))l(k)^2$,
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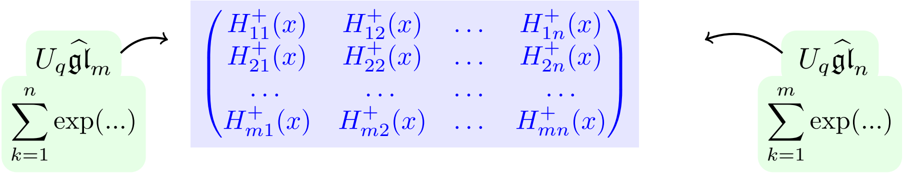

$U_q\widehat{\mathfrak{gl}}_m$ $\sum\limits_{k=1}^{n} \exp(...)$

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Lemma. ([FJM]) *We have* $[U_q\widehat{\mathfrak{gl}}_m, U_q\widehat{\mathfrak{gl}}_n] = 0$ *in* $\operatorname{End}(V)$.

The duality of integrable systems

Choose evaluation parameters

$$U_q\widehat{\mathfrak{gl}}_m \curvearrowright \begin{pmatrix} H^+_{11}(x) & H^+_{12}(x) & \dots & H^+_{1n}(x) \\ H^+_{21}(x) & H^+_{22}(x) & \dots & H^+_{2n}(x) \\ \dots & \dots & \dots & \dots \\ H^+_{m1}(x) & H^+_{m2}(x) & \dots & H^+_{mn}(x) \end{pmatrix} \curvearrowleft U_q\widehat{\mathfrak{gl}}_n$$

The duality of integrable systems

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$$U_q\widehat{\mathfrak{gl}}_m \curvearrowright \begin{array}{ccccc} z_1 & z_2 & \dots & z_n & \\ \left(\begin{matrix} H^+_{11}(x) & H^+_{12}(x) & \dots & H^+_{1n}(x) \\ H^+_{21}(x) & H^+_{22}(x) & \dots & H^+_{2n}(x) \\ \dots & \dots & \dots & \dots \\ H^+_{m1}(x) & H^+_{m2}(x) & \dots & H^+_{mn}(x) \end{matrix}\right) & & & & \begin{matrix} u_1 \\ u_2 \\ \dots \\ u_m \end{matrix} \end{array} \curvearrowleft U_q\widehat{\mathfrak{gl}}_n$$

The duality of integrable systems

Choose evaluation parameters, choose q_1, $q_1^\vee$.

$$\begin{tikzcd}
U_q\widehat{\mathfrak{gl}}_m \arrow[r] & \begin{array}{cc} \begin{array}{cccc} z_1 & z_2 & \dots & z_n \end{array} & \\ \begin{pmatrix} H^+_{11}(x) & H^+_{12}(x) & \dots & H^+_{1n}(x) \\ H^+_{21}(x) & H^+_{22}(x) & \dots & H^+_{2n}(x) \\ \dots & \dots & \dots & \dots \\ H^+_{m1}(x) & H^+_{m2}(x) & \dots & H^+_{mn}(x) \end{pmatrix} & \begin{array}{c} u_1 \\ u_2 \\ \dots \\ u_m \end{array} \end{array} & U_q\widehat{\mathfrak{gl}}_n \arrow[l]
\end{tikzcd}$$

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Choose evaluation parameters, choose q_1, $q_1^\vee$. Then one can define:

$$\mathcal{E}_m(q_1,q) \longrightarrow \begin{array}{ccccc} z_1 & z_2 & \dots & z_n & \\ H^+_{11}(x) & H^+_{12}(x) & \dots & H^+_{1n}(x) & u_1 \\ H^+_{21}(x) & H^+_{22}(x) & \dots & H^+_{2n}(x) & u_2 \\ \dots & \dots & \dots & \dots & \dots \\ H^+_{m1}(x) & H^+_{m2}(x) & \dots & H^+_{mn}(x) & u_m \end{array} \longleftarrow \mathcal{E}_n(q_1^\vee,q)$$

The duality of integrable systems

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$$\mathcal{E}_m(q_1,q) \curvearrowright \begin{array}{cccccc} & z_1 & z_2 & \cdots & z_n & \\ \left(\begin{array}{c} \\ \\ \\ \\ \end{array}\right. & \begin{array}{c} H^+_{11}(x) \\ H^+_{21}(x) \\ \cdots \\ H^+_{m1}(x) \end{array} & \begin{array}{c} H^+_{12}(x) \\ H^+_{22}(x) \\ \cdots \\ H^+_{m2}(x) \end{array} & \begin{array}{c} \dots \\ \dots \\ \dots \\ \dots \end{array} & \begin{array}{c} H^+_{1n}(x) \\ H^+_{2n}(x) \\ \cdots \\ H^+_{mn}(x) \end{array} & \left.\begin{array}{c} \\ \\ \\ \\ \end{array}\right) \begin{array}{c} u_1 \\ u_2 \\ \cdots \\ u_m \end{array} \end{array} \curvearrowleft \mathcal{E}_n(q_1^\vee,q)$$

Lemma. *As an $\mathcal{E}_m(q_1,q)$ module,*

$$V = \bigoplus_{k_1,\dots,k_n=0}^{\infty} \mathcal{F}^{(m)}_{r(k_1)}(u_1(k_1),t(k_1),k_1) \otimes \cdots \otimes \mathcal{F}^{(m)}_{r(k_n)}(u_n(k_n),t(k_n),k_n).$$

Here $u(k) = (-1)^m (q_1 q)^{-k-m/2} q u.$

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Here $u(k) = (-1)^m (q_1 q)^{-k-m/2} qu.$

Theorem. ([FJM]) *We have* $[\widehat{\mathcal{B}}^{p}_m, \widehat{\mathcal{B}}^{p^\vee}_n] = 0$ *in* $\mathrm{End}(V)$ *provided*

$$p_i = u_{i+1}/u_i, \quad p_i^\vee = z_{i+1}/z_i, \quad p_0 = (q_1^\vee)^n, \quad p_0^\vee = q_1^m.$$

Conformal limit

Let $m = 1, n = 2$. We have $\mathcal{E}_1(q_1, q)$ and $\mathcal{E}_2(q_1^\vee, q)$ acting on a two boson space. Set $q = 1 - \epsilon/2 + o(\epsilon)$, and then

$$\begin{aligned} q_1 &= 1 + (1-r)\epsilon + o(\epsilon), & z_1/z_2 &= 1 - \kappa\epsilon + o(\epsilon), & p_0 &= e^\tau(1 + o(\epsilon)), \\ q_1^\vee &= e^{-\tau}(1 + \epsilon + o(\epsilon)), & p_0^\vee &= 1 + r\epsilon + o(\epsilon), & p_1^\vee &= 1 - \kappa\epsilon/2 + o(\epsilon). \end{aligned}$$

The limit $\epsilon \to 0$ is called Intermediate Long Wave limit.
Further limit $\tau \to 0$ is called conformal limit.

In the conformal limit:

- one of the two bosons commutes with all operators in the theory and can be factored out;
- the current $F(x)$ of $\mathcal{E}_1(q_1, q)$ is identified to the Virasoro current $T(z)$;
- the remaining boson is identified with Virasoro Verma module of central charge $c = 1 - 6(1-\beta)^2/\beta$ and highest weight $h = (\kappa^2 - 1)(1-\beta)^2/(4\beta)$, where $\beta = (r-1)/r$.

Spectrum of local integrals of motion

The Virasoro algebra has an algebra of quantum Hamiltonians called local integrals of motion, [FF] also known as quantum KdV flows.

The first non-trivial local integral of motion is $I_2 = \int :T(x)^2: dx/x$.

Theorem. ([FKSW], [FJM1]) *The conformal limit of $\widehat{\mathcal{B}}_1^p$ coincides with the algebra of local integrals of motion.*

It is known that spectrum of $\widehat{\mathcal{B}}_1^p$ is given by Bethe ansatz [FJMM1], [FJMM2]. This gives the conjecture of [L]:

Theorem. ([FJM1]) The spectrum of local integral of motion is described by the solutions of Bethe ansatz equation:

$$\frac{t_i}{t_i-1}\,\frac{t_i-\kappa}{t_i-\kappa-1}\prod_{j=1}^{N}\frac{t_i-t_j-1}{t_i-t_j+1}\,\frac{t_i-t_j+r}{t_i-t_j-r}\,\frac{t_i-t_j-r+1}{t_i-t_j+r-1}=-1.$$

This is double Yangian (XXX type) Bethe ansatz equation associated to $\mathfrak{gl}_1$.
This description is different from the one suggested in [BLZ].

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B. Feigin, M. Jimbo, T. Miwa, and E.M. (15,16)

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Spectrum of non-local integrals of motion

One can also define non-local integrals of motion, [BLZ1]. The non-local integrals of motion are given by integrals of products of vertex operators.

Conjecture. ([FKSW], [FJM1]) *The conformal limit of $\widehat{\mathcal{B}}_2^{p^\vee}$ coincides with the algebra of non-local integrals of motion.*

The spectrum of $\widehat{\mathcal{B}}_2^{p^\vee}$ is also given by Bethe ansatz.

Conjecture. ([FJM2]) The spectrum of non-local integrals of motion is described by the solutions of Bethe ansatz equation:

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Questions?

Thank you!