Random matrices and history dependent stochastic processes.

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- History dependent stochastic processes

  memory effects, self-learning...

  ex: ants looking for the best route nest-food

- Lattice random Schrödinger operators

  quantum diffusion for disordered materials

These subjects are connected!

Example: linearly edge-reinforced random walk (ERRW) (Diaconis 1986)

discrete time process $(X_n)_{n\geq 0}$, $X_n \in \mathbb{Z}^d$ or $\Lambda \subset\subset \mathbb{Z}^d$

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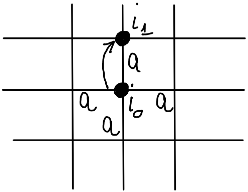

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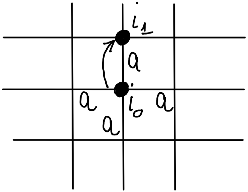

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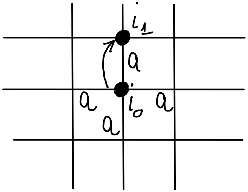

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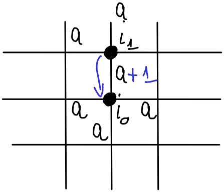

- set $X_0 = i_0, X_1 = i_1$,

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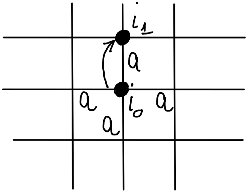

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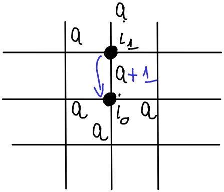

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prefers to come back!

after n steps

$$\mathbb{P}(X_{n+1} = j | X_n = i, (X_m)_{m \le n}) = \mathbf{1}_{|i-j|=1} \frac{\omega_{ij}(n)}{\sum_{k,|k-j|=1} \omega_{ik}(n)}$$

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a reinforcement parameter

the first time e is crossed

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Generalizations

- Λ any locally finite graph
- variable initial weights a_e

Vertex-reinforced jump process (VRJP)

(Werner 2000, Volkov, Davis)

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Connections with

- **ERRW** [Sabot-Tarrès 2013]
- hitting times for interacting Brownian motions
- nonlinear sigma models and statistical mechanics
- **random matrices** [Sabot-Tarrès-Zeng 2015] [Sabot-Zeng 2015]

transience/recurrence for VRJP and ERRW as $\Lambda \to \mathbb{Z}^d$

positive recurrence

- at strong reinforcement: ERRW and VRJP **for any** $d \geq 1$

[Merkl-Rolles 2009], [D.-Spencer 2010] [Sabot-Tarrès 2013]

[Angel-Crawford-Kozma.Angel 2014]

- for any reinforcement: ERRW and VRJP in $d = 1$ and strips

[Merkl-Rolles 2009], [D.-Spencer 2010] [Sabot-Tarrès 2013] [D.-Merkl-Rolles 2014]

recurrence in $d = 2$

- ERRW for **any** reinforcement, partial results for VRJP

[Merkl-Rolles 2009], [Sabot-Zeng 2015] [Bauerschmidt-Helmuth-Swan 2018]

transience in $d \geq 3$

- at weak reinforcement: ERRW and VRJP

[D.-Spencer-Zirnbauer 2010], [D.-Sabot-Tarrès 2015]

$\Rightarrow$ phase transition in $d \geq 3$

Random matrices

- set up: $\Lambda \subset \mathbb{Z}^d$ **finite**, $H_\Lambda \in \mathbb{C}^{\Lambda \times \Lambda}$
  - $H_\Lambda^* = H_\Lambda$
  - H_Λ random with some probability $d\mathbb{P}_\Lambda(H)$

Question: $\lim_{\Lambda \to \mathbb{Z}^d} d\mathbb{P}_\Lambda(H) = ?$
spectral properties of the limit operator?

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- special case: random Schrödinger $H_\Lambda = -\Delta_\Lambda + \lambda\hat{V}$

[Anderson 1958]

  - $-\Delta_\Lambda$ lattice Laplacian, $\lambda > 0$ parameter
  - $\hat{V} = \mathrm{diag}(\{V_x\}_{x\in\Lambda})$, $V \in \mathbb{R}^\Lambda$ random vector $d\mathbb{P}_\Lambda(V)$

motivation: quantum mechanics, disordered conductors two limit cases:

- $\lambda = 0 : H = -\Delta : l^2(\mathbb{Z}^d) \to l^2(\mathbb{Z}^d)$
  extended states: H has only generalized eigenfunctions
  $\psi_{\lambda(k)}(x) = e^{ik\cdot x} \notin l^2(\mathbb{Z}^d)$ conductor
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results/conjectures in $d \geq 2$

Assume V independent or short range correlated:

- large disorder $\lambda \gg 1$: exponentially localized eigenfunctions $\forall d \geq 2$

  [Fröhlich-Spencer 1983], [Aizenman-Molchanov 1993] and many other results later. . .

- $d = 2$ exponentially localized eigenfunctions $\forall \lambda$ (conjecture)
- $d \geq 3$ phase transition at weak disorder (conjecture)

A special example of random Schrödinger operator: $H_W(\beta) := 2\hat{\beta} - WP$

- $-P = -\Delta - 2d\mathrm{Id}$ (off-set Laplacian) $P_{ij} = \mathbf{1}_{|i-j|=1}$
- $\beta \in \mathbb{R}^{\Lambda}$ random vector with distribution

$$\mathbf{1}_{H(\beta)>0} \left(\frac{2}{\pi}\right)^{|\Lambda|/2} e^{W2d|\Lambda|} e^{-\sum_{j\in\Lambda}\beta_j} \frac{1}{(\det H_W(\beta))^{\frac{1}{2}}} d\beta_\Lambda$$

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- for **wired** boundary conditions $\lim_{\Lambda\to\mathbb{Z}^d} d\mathbb{P}_\Lambda(\beta)$ exists
- $H_W(\beta) \equiv -\Delta + \lambda\hat{V}$ with $\lambda = \frac{1}{W}$:

$$H_W(\beta) = W(-\Delta + \tfrac{1}{W}\hat{V}), \qquad V_x = 2\beta_x - 2dW$$

$$\mathbb{E}[V_x] = 2\mathbb{E}[\beta_x] - 2dW = (2dW+1) - 2dW = 1.$$

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- $\mathcal{L}_{u,W} = e^{\hat{u}} H_W(\beta(u)) e^{-\hat{u}}$ with $\beta_x(u) = \sum_{y,|y-x|=1} \frac{W}{2} e^{u_y - u_x}$

$$d\mathbb{P}(u) \to d\mathbb{P}(\beta) \text{ coordinate change!}$$

additional nice features

- $\frac{1}{W} = \lambda \Rightarrow$ strong/weak reinforcement $\equiv$ strong/weak disorder
- ground state for $H_W(\beta) \longleftrightarrow$ recurrence/transience for VRJP
  [Sabot-Zeng 2015]
- β short range $\Rightarrow$ standard fractional moment methods for RS apply [Collevecchio-Zeng 2018]
- still a lot to explore!

THANK YOU