On multiple SLEs

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Based on joint works with

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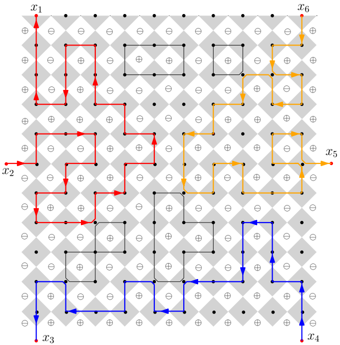
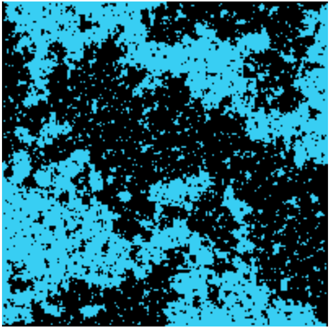
ICMP, Equilibrium Statistical Mechanics Session

PLAN

1. Motivation
   - critical models in statistical physics, scaling limits
   - conformal invariance of interfaces & correlations
2. Multiple SLEs: classification
   - "global" multiple SLEs
   - "local" multiple SLEs (i.e., commuting SLEs)
3. Partition functions of multiple SLEs
   - marginals of global multiple SLEs
   - local $\Leftrightarrow$ global ???
4. Relation to connection probabilities
   - multichordal loop-erased random walks / UST branches
   - level lines of the Gaussian free field
   - double-dimer pairings
   - Ising model crossing probabilities

Motivation



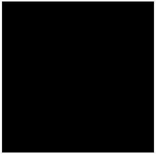


Example: 2D Ising model

- put random spins $\sigma_x = \pm 1$ at vertices x of a graph
- nearest neighbor interaction: $\mathbb{P}\,[\text{config.}] \propto \exp\left(\frac{1}{T} \sum_{x\sim y} \sigma_x \sigma_y\right)$
- **phase transition** at critical temperature $T = T_c$

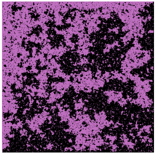
look at correlation of a pair of spins at x and y

$C(x, y) = \mathbb{E}\,[\sigma_x \sigma_y] - \mathbb{E}\,[\sigma_x]\,\mathbb{E}\,[\sigma_y]$ when $|x - y| >> 1$:



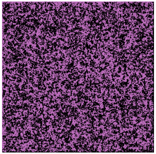
$T < T_c$

$C(x, y) \sim \text{const.}$



$T = T_c$

$C(x, y) \sim |x - y|^{-\beta}$

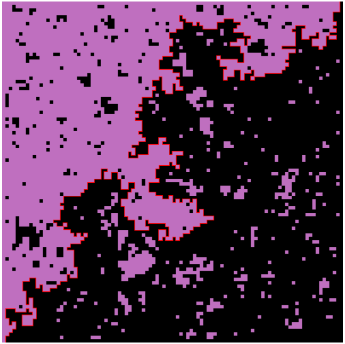


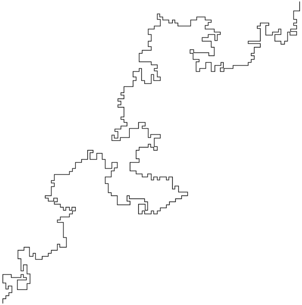
$T_c < T$

$C(x, y) \sim e^{-\frac{1}{\xi}|x-y|}$

- scaling limit at *critical temperature* T_c: **conformal invariance**

Dobrushin boundary conditions: $\partial\Omega^\delta = \{\oplus \text{ segment}\} \bigcup \{\ominus \text{ segment}\}$

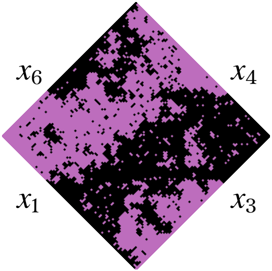




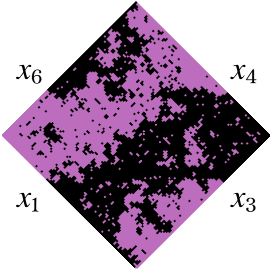
interface of Ising model $\xrightarrow{\delta\to 0}$ Schramm-Loewner evolution, SLE_3

[Chelkak, Duminil-Copin, Hongler, Kemppainen, Smirnov (2014)]

Conformal invariance of multiple Ising interfaces



- fix discrete domain data $(\Omega^\delta; x_1^\delta, \ldots, x_{2N}^\delta)$
- consider critical **Ising model** in $\Omega^\delta \subset \delta\mathbb{Z}^2$ with alternating $\oplus/\ominus$ b.c.
- **Izyurov (2015):**
  interfaces $\xrightarrow{\delta \to 0}$ (local) multiple SLE_3
  Proof: multi-point holomorphic observable



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- If we condition on the event that the interfaces connect the boundary points according to a given connectivity ...

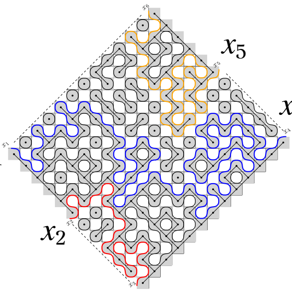
... then we have:

Theorem

The law of the N macroscopic interfaces of the critical Ising model **converges in the scaling limit $\delta \to 0$ to the N-SLE_κ with $\kappa = 3$.**

Wu **[arXiv:1703.02022]**
Beffara, P. & Wu **[arXiv:1801.07699]**

Proof: convergence for $N = 1$ and classification of multiple SLE_3

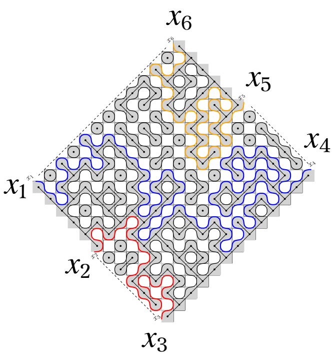


- fix discrete domain data $(\Omega^\delta; x_1^\delta, \ldots, x_{2N}^\delta)$
- consider critical **FK-Ising model** in $\Omega^\delta \subset \delta\mathbb{Z}^2$ with alternating free/wir b.c.
- **Kemppainen & Smirnov (2018):**

2 interfaces $\xrightarrow{\delta\to 0}$ $\mathrm{hSLE}_{16/3}$

Proof: holomorphic observable

Conformal invariance of multiple FK-Ising interfaces



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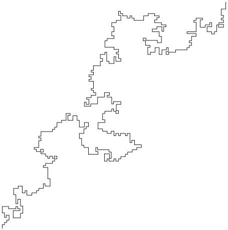
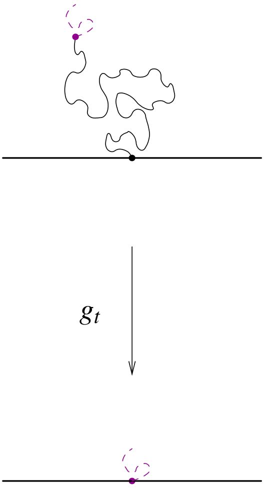


Schramm-Loewner

EVOLUTIONS

SLE_κ

Schramm's classification of chordal SLE$_\kappa$



$X_t = g_t(\gamma(t))$

Theorem [Schramm ~2000]

$\exists!$ one-parameter family $(\mathrm{SLE}_\kappa)_{\kappa \geq 0}$ of probability measures on chordal curves with **conformal invariance** and **domain Markov property**

- encode SLE$_\kappa$ random curves in **random conformal maps** $(g_t)_{t \geq 0}$
- driving process = **image of the tip**:

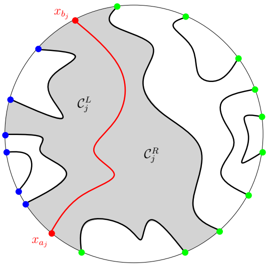
$$X_t := \lim_{z \to \gamma(t)} g_t(z) = \sqrt{\kappa} B_t$$

- $g_t : \mathbb{H} \setminus \gamma[0,t] \to \mathbb{H}$ solutions to **Loewner equation**:

$$\frac{\mathrm{d}}{\mathrm{d}t} g_t(z) = \frac{2}{g_t(z) - X_t}, \qquad g_0(z) = z$$

Classification of global multiple SLE$_\kappa$

- family of **random curves** in $(\Omega; x_1, \ldots, x_{2N})$
- various connectivities encoded in **planar pair partitions** $\alpha \in \mathrm{LP}_N$



Theorem

Let $\kappa \in (0,4] \cup \{16/3, 6\}$. For any fixed connectivity α of $2N$ points, **there exists a unique probability measure on N curves** such that *conditionally on $N-1$ of the curves, the remaining one is the chordal* SLE$_\kappa$ *in the random domain where it can live.*

Dubédat (2006); Kozdron & Lawler (2007–2009); Miller & Sheffield (2016); Miller, Sheffield & Werner (2018); P. & Wu (2017); Beffara, P. & Wu (2018)

Theorem

Let $\kappa \in (0, 4] \cup \{16/3, 6\}$. For any fixed connectivity α of $2N$ points, **there exists at most one probability measure on N curves**

such that

conditionally on $N - 1$ of the curves, the remaining one is the chordal SLE_κ in the random domain where it can live.

Idea of proof: [Beffara, P. & Wu **[arXiv:1801.07699]**]

- sample curves according to conditional law
  $\Rightarrow$ **Markov chain** on space of curves
- prove that there is a coupling of two such Markov chains, started from *any* two initial configurations, such that they have a *uniformly positive chance to agree after a few steps*
  $\Rightarrow$ **there is *at most one* stationary measure**

Uniqueness: $\exists$ at most one global N-SLE_κ

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Remarks:

- uniformly exponential mixing: $\exists$ coupling s.t. $\mathbb{P}[X_{4n} \neq \tilde{X}_{4n}] \leq (1-p_0)^n$

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- update: $N=2$, $\kappa \in (0,8)$: Miller, Sheffield, Werner (2018)

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Marginal law of one curve

Theorem:

Let $\kappa \in (0,4] \cup \{16/3, 6\}$. For any fixed connectivity α of $2N$ points, **there exists a unique probability measure on N curves** s.t. *conditionally on $N-1$ of the curves, the remaining one is the chordal* SLE_κ *in the random domain where it can live.*

Proposition

Let $\kappa \in (0,4]$. The marginal law of the curve starting from x_1 is given by the Loewner chain with driving process

$$\mathrm{d}W_t = \sqrt{\kappa}\,\mathrm{d}B_t + \kappa\,\partial_1 \log \mathcal{Z}_\alpha\left(W_t, V_t^2, V_t^3, \ldots, V_t^{2N}\right)\mathrm{d}t, \qquad W_0 = x_1$$

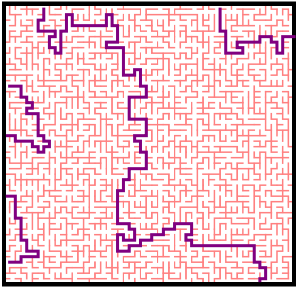
$$\mathrm{d}V_t^i = \frac{2\mathrm{d}t}{V_t^i - W_t}, \qquad V_0^i = x_i, \quad \text{for } i \neq 1$$

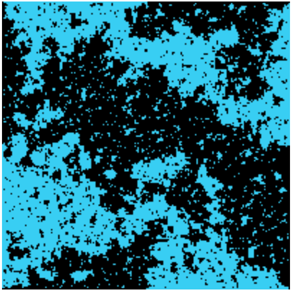
Therefore, **local** N-SLE_κ with partition function $\mathcal{Z}_\alpha$

= **global** N-SLE_κ associated to connectivity α

P. & Wu **[arXiv:1703.00898]**

Connection probabilities



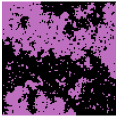
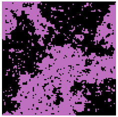
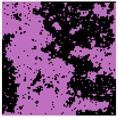
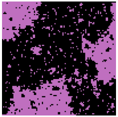
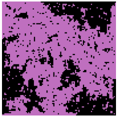


Application: connection probabilities

Idea: discrete connection probabilities $\xrightarrow{\delta\to 0}$ partition functions:

$$\lim_{\delta\to 0} \mathbb{P}\,[\text{interfaces form connectivity } \alpha] = \frac{\mathcal{Z}_\alpha(x_1,\ldots,x_{2N})}{\sum_\beta \mathcal{Z}_\beta(x_1,\ldots,x_{2N})}$$

[Suggested e.g. by Bauer, Bernard, Kytölä (2005)]

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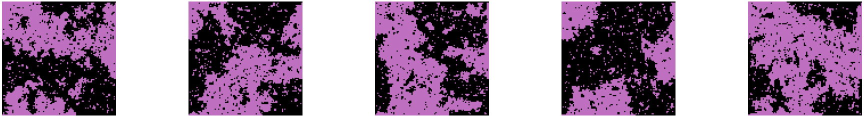
- **Gaussian free field** ($\kappa = 4$) with alternating boundary data $+\lambda, -\lambda, +\lambda, -\lambda, \ldots$ [P. & Wu **[arXiv:1703.00898]**]
- **double-dimer model** ($\kappa = 4$) [Kenyon & Wilson (2011)]
- boundary touching branches in **uniform spanning tree** ($\kappa = 2$) with wired boundary [Karrila, Kytölä, P. **[arXiv:1702.03261]**]
- **critical Ising model** with alternating b.c. [in progress...]




Thanks !