The KPZ fixed point

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joint work with

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1d randomly stirred fluid: Stochastic Burgers equation

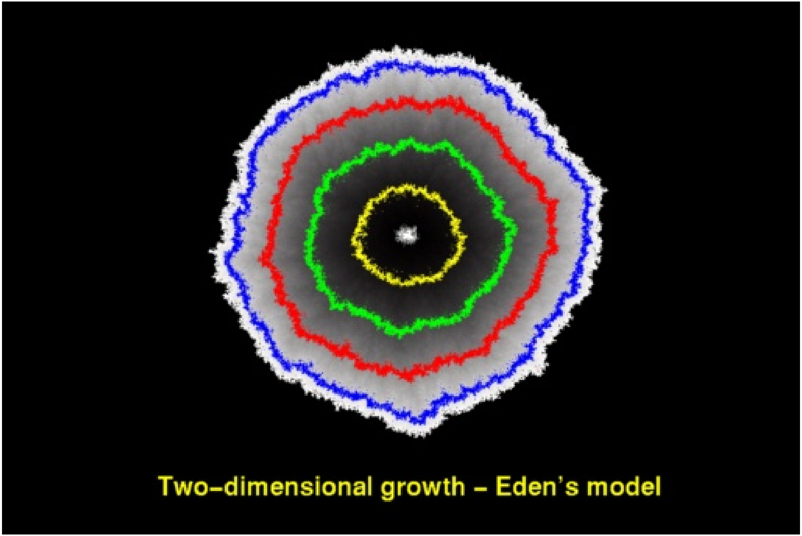
$$\partial_t u = \partial_x u^2 + \partial_x^2 u + \partial_x \xi \qquad t > 0, \quad x \in \mathbb{R}$$

$\xi(t, x)$ space-time white noise

Forster-Nelson-Stephen 77: Dynamic renormalization group

$\epsilon^{-1/2} u(\epsilon^{-3/2} t, \epsilon^{-1} x) \quad \rightarrow$ Non-linear long time fixed pt

Anomalous fluctuations $\rightarrow$ A new universality class ?

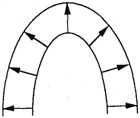


$$t(\text{some shape}) + t^{\chi}\text{Fluctuations}$$

$1+1$ d: Fluctuation exponent $\chi = 1/3$

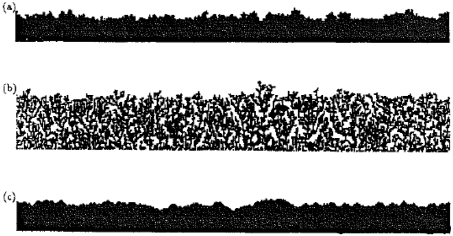
Kardar-Parisi-Zhang (KPZ) Equation 86

$$\partial_t h = \underbrace{(\partial_x h)^2}_{\text{lateral growth}} + \underbrace{\partial_x^2 h}_{\text{relaxation}} + \underbrace{\xi}_{\text{space-time white noise}} \qquad \underbrace{h(t,x)}_{\text{height at time } t \text{ position } x}$$



$\partial_x h = u \rightsquigarrow$ stoch Burgers eqn
$\partial_t u = 2u\partial_x u + \partial_x^2 u + \partial_x \xi$

T. Halpin-Healy, Y.-C. Zhang / Physics Reports 254 (1995) 215–414

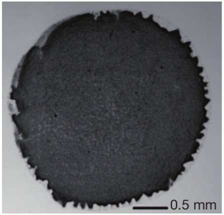


Eden growth

Ballistic aggregration

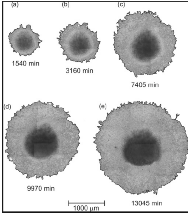
ASEP

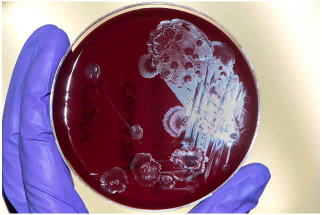
2.2. Snapshots illustrating surface and bulk properties of three distinct stochastic growth models: (a) Eden cluster, (b) stic deposit, and (c) RSOS solid. All belong to the KPZ universality class [HH93].



Burning paper

Coffee Stains



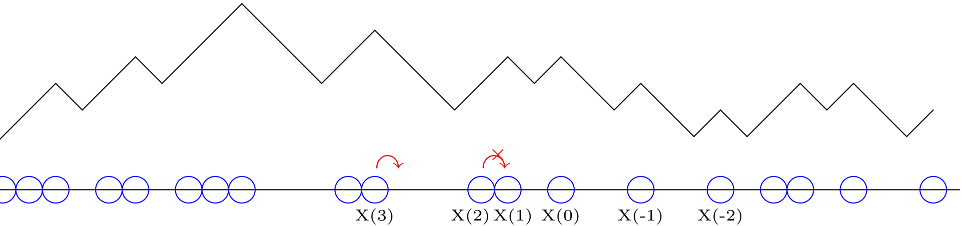


Tumour growth (?)

Bacterial growth

A special discretization of KPZ equation (TASEP)

$h(x+1) = h(x) \pm 1$, $x \in \mathbb{Z}$
local max $\mapsto$ local min at rate 1



$$-2\mathbf{1}_\wedge = \frac{1}{2}\left[(\nabla^- h)(\nabla^+ h) - 1 + \Delta h\right]$$

symmetric random walk invariant (except for height shift)

Lab mouse of non-equilibrium stat mech since the late 60's

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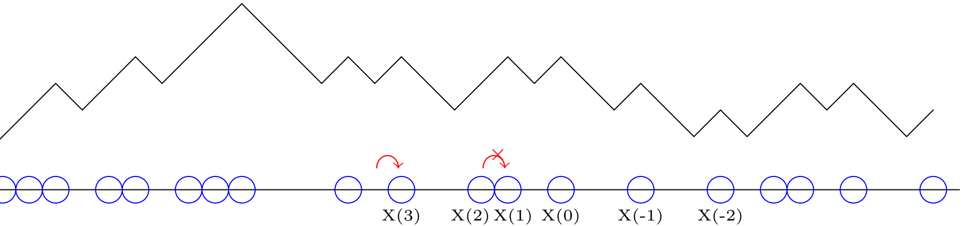

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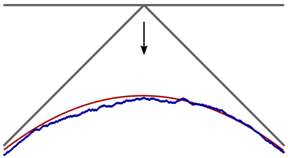
Asymptotic fluctuations depend on initial data

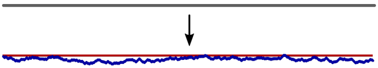
In TASEP special initial data could be computed 00's Johansson, Spohn, Borodin, Sasamoto,...

Corner



$$h(t,x) \sim c_1 t - c_2 \frac{x^2}{t} + c_3 t^{1/3} F_{\mathrm{GUE}}$$

Flat



$$h(t,x) \sim c_3 t + c_4 t^{1/3} F_{\mathrm{GOE}}$$

$F_{\mathrm{GUE}}/F_{\mathrm{GOE}}$ are the rescaled top eigenvalues of a matrix from the Gaussian Unitary/Orthogonal Ensembles

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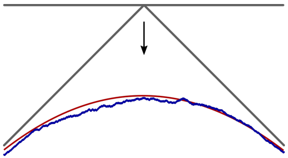

$$h(t,x) \sim c_1 t - c_2 \frac{x^2}{t} + c_3 t^{1/3} \mathcal{A}_2(t^{-2/3}x)$$

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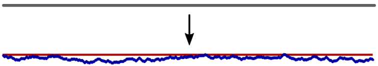
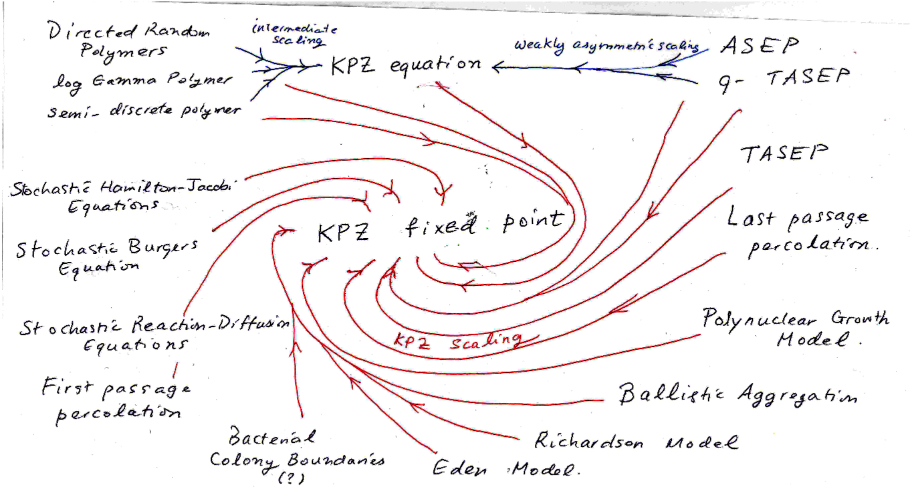


$$h(t,x) \sim c_3 t + c_4 t^{1/3} \mathcal{A}_1(t^{-2/3}x)$$

$\mathrm{Airy}_2/\mathrm{Airy}_1$ are special stochastic processes

Equivalent to $h_\epsilon(t,x) = \epsilon^{1/2} h(\epsilon^{-3/2}t, \epsilon^{-1}x) - c\epsilon^{-3/2}t$

Conjectural KPZ universality class



KPZ fixed point is space-time random field at the centre, the non-linear fixed point invariant under $h_\epsilon(t,x) = \epsilon^{1/2} h(\epsilon^{-3/2} t, \epsilon^{-1} x)$

KPZ eq limit of $\epsilon^{1/2} h(\epsilon^{-2} t, \epsilon^{-1} x)$ with nonlinearity/noise of order $\epsilon^{1/2}$

What is the fixed point?

- Formally, solution of Hopf equation $\partial_t \mathfrak{h} = (\partial_x \mathfrak{h})^2$, or $\partial_x \mathfrak{h} = \mathfrak{u}$ solves Burgers equation $\partial_t \mathfrak{u} = \partial_x \mathfrak{u}^2$, but *they are not well posed*
- Instead, we describe it as a Markov process whose transition probabilities can be linearized, an *integrable Markov process*
- This is done by first solving TASEP, which also turns out to be an *integrable Markov process*, then taking limit
- At TASEP level there are well-posed equations (the Kolmogorov or backward equation, or the Fokker-Planck or forward equation) so the solution can be checked
- Method works for some variants of TASEP, but the fact that there is only one such fixed point is still conjectural

The KPZ fixed point

We give formulas for fixed point; formulas for TASEP similar with Brownian motion replaced by random walk

Markov transition probabilities are given by determinants

$$\mathbf{Prob}\big(\mathfrak{h}(\boldsymbol{t},\mathbf{x}_1) \le \mathbf{a}_1,\ldots,\mathfrak{h}(\boldsymbol{t},\mathbf{x}_M) \le \mathbf{a}_M \mid \mathfrak{h}(0,\cdot) = \mathfrak{h}_0(\cdot)\big) = \det(\mathbf{I} - \mathbf{K}_{\mathfrak{h}_0,\boldsymbol{t},\mathbf{x},\mathbf{a}})$$

Fredholm determinant of a compact (trace class) operator

$$\det(I+K)_{L^2(S,\mu)} = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{S^n} \det\left[K(u_i,u_j)\right]_{i,j=1}^{n} d\mu(u_1)\cdots d\mu(u_n)$$

Stochastic integrable system: dynamics linearized by a novel *Brownian scattering transform* $\mathfrak{h} \mapsto \mathbf{K}_{\mathfrak{h}}$

Brownian scattering transform

For $\mathfrak{h}$ upper semi-continuous define for **B** Brownian motion

$$\mathbf{P}^{\text{hit}\,\mathfrak{h}}_{-L,L}(u_1, u_2) = \mathbb{P}_{\mathbf{B}(-L)=u_1,\mathbf{B}(L)=u_2}\Big(\mathbf{B} \text{ hits } \text{hypo}(\mathfrak{h}) \text{ on } [-L, L]\Big)$$

The Brownian scattering transform of $\mathfrak{h}$ is

$$\mathbf{K}_{\mathfrak{h}} = \lim_{L\to\infty} e^{-L\partial^2}\mathbf{P}^{\text{hit}\,\mathfrak{h}}_{-L,L}e^{-L\partial^2}$$

Looks terrible because of backwards heat equation, but we only ever use

$$\mathbf{K}_{\mathfrak{h},\boldsymbol{t}} = U_{\boldsymbol{t}}\mathbf{K}^{\mathfrak{h}_0}U_{\boldsymbol{t}}^{-1} \qquad U_{\boldsymbol{t}} = e^{\frac{\boldsymbol{t}}{3}\partial^3}$$

Ok! $e^{\mathbf{x}\partial^2+\frac{\boldsymbol{t}}{3}\partial^3}(z_1, z_2) = \boldsymbol{t}^{-1/3}e^{\frac{2\mathbf{x}^3}{3\boldsymbol{t}^2}-\frac{(z_1-z_2)\mathbf{x}}{\boldsymbol{t}}}\,\mathrm{Ai}(-\boldsymbol{t}^{-1/3}(z_1-z_2)+\boldsymbol{t}^{-4/3}\mathbf{x}^2)$
even for $x < 0$ as long as $t \neq 0$!

KPZ fixed pt formula

$$\mathbb{P}_{\mathfrak{h}_0}\big(\mathfrak{h}(\boldsymbol{t},\mathbf{x}_1)\le \mathbf{a}_1,\dots,\mathfrak{h}(\boldsymbol{t},\mathbf{x}_M)\le \mathbf{a}_M\big)=\det\big(\mathbf{I}-\chi_{\mathbf{a}}U_{\boldsymbol{t}}\mathbf{K}^{\mathrm{ext}}_{\mathfrak{h}_0}U_{\boldsymbol{t}}^{-1}\chi_{\mathbf{a}}\big)$$

$$\mathbf{K}^{\mathrm{ext}}_{\mathfrak{h}}(\mathbf{x}_i,\cdot;\mathbf{x}_j,\cdot)=-e^{(\mathbf{x}_j-\mathbf{x}_i)\partial^2}\mathbb{1}_{\mathbf{x}_i<\mathbf{x}_j}+e^{-\mathbf{x}_i\partial^2}\mathbf{K}^{\mathfrak{h}_0}e^{\mathbf{x}_j\partial^2} \qquad \chi_{\mathbf{a}}(\mathbf{x}_i,u)=\mathbb{1}_{u>\mathbf{a}_i}$$

- Conjecturally unique local dynamics satisfying
  1. (1:2:3 scaling invariant) $\alpha\mathfrak{h}(\alpha^{-3}\boldsymbol{t},\alpha^{-2}\mathbf{x};\alpha\mathfrak{h}_0(\alpha^{-2}\mathbf{x}))\overset{dist}{=}\mathfrak{h}(\boldsymbol{t},\mathbf{x};\mathfrak{h}_0)$
  2. (Skew time reversible) $\mathbb{P}\big(\mathfrak{h}(\boldsymbol{t},\mathbf{x};\mathfrak{g})\le-\mathfrak{f}(\mathbf{x})\big)=\mathbb{P}\big(\mathfrak{h}(\boldsymbol{t},\mathbf{x};\mathfrak{f})\le-\mathfrak{g}(\mathbf{x})\big)$
  3. (Stationarity in space) $\mathfrak{h}(\boldsymbol{t},\mathbf{x}+\mathbf{u};\mathfrak{h}_0(\mathbf{x}-\mathbf{u}))\overset{dist}{=}\mathfrak{h}(\boldsymbol{t},\mathbf{x};\mathfrak{h}_0)$;
- For TASEP (and a few related models) if the rescaled height function converges to $\mathfrak{h}_0$ as upper semicont fns
  $\epsilon^{1/2}\Big[h(2\epsilon^{-3/2}\boldsymbol{t},2\epsilon^{-1}\mathbf{x})+\epsilon^{-3/2}\boldsymbol{t}\Big]\xrightarrow[\epsilon\to0]{}\mathfrak{h}(\boldsymbol{t},\mathbf{x};\mathfrak{h}_0)$ in distr
- For $\boldsymbol{t}>0$, $\mathfrak{h}(\boldsymbol{t},\mathbf{x})$ locally Brownian, Hölder $\frac{1}{2}-$ in $\mathbf{x}$, $\frac{1}{3}-$ in $\boldsymbol{t}$
- For eg, flat, narrow wedge, Brownian scattering transform is easy to compute and recovers the known formulas for the Airy processes.

Biorthogonalization of TASEP

One-sided i.d. $X(0) = X(-1) = \cdots = +\infty$, $1 \leq n_1 < n_2 < \cdots < n_M$

Borodin, Ferrari, Prähofer and Sasamoto'07:

$$\mathbb{P}_{X_0}\Big(X_t(n_j) > a_j,\ j = 1, \ldots, M\Big) = \det\Big(I - \mathbb{1}_{x_i \leq a_i} K_t \mathbb{1}_{x_i \leq a_i}\Big)_{\ell^2(\{n_1,\ldots,n_M\}\times\mathbb{Z})}$$

$$K_t(n_i, x_i; n_j, x_j) = -Q^{n_j - n_i}(x_i, x_j)\mathbb{1}_{n_i < n_j} + \sum_{k=1}^{n_j} \Psi^{n_i}_{n_i-k}(x_i)\,\Phi^{n_j}_{n_j-k}(x_j)$$

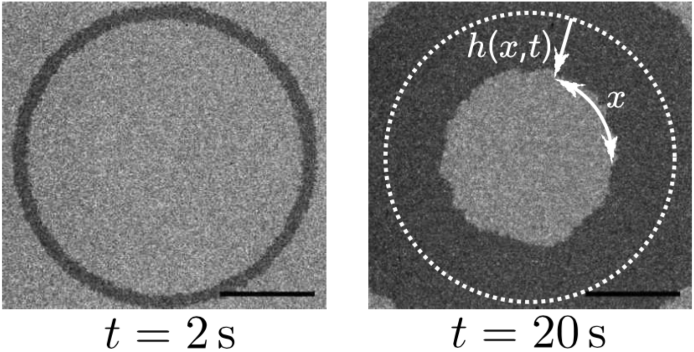
$$Q(x, y) = \mathbb{1}_{x>y} \qquad \Psi^n_k(x) = \frac{1}{2\pi\mathrm{i}} \oint_{\Gamma_0} \frac{dw}{w^{x - X_0(n-k)+1}} \left(\frac{1-w}{w}\right)^k e^{t(w-1/2)}$$

Ψ^n_k Charlier polynomials. Φ^n_k are defined implicitly by

1. Biorthogonality: $\sum_{x\in\mathbb{Z}} \Psi^n_\ell(x)\Phi^n_k(x) = \mathbb{1}_{\ell=k}$
2. Φ^n_k is a polynomial of degree k

Φ^n_k were only known for particles equally spaced in a block. We find them in general

Exact fluctuations starting from $\mathfrak{h}_0(x) = x^2$ explains recent experimental results of Takeuchi on fluctuations of inward growing fronts (incorrect claims that it has flat fluctuations, F_{GOE})



Here we have finite time ($t = 50s$) blowup. If initial data has (say) linear growth at ∞ then solutions for all time

Conclusions

- At least now we can state the

  KPZ Universality conjecture: All models in the class should converge to this fixed point under the 1:2:3 scaling

  We have proved it now for TASEP and several generalizations

- The solution of TASEP (after 50 yrs!) has ramifications well beyond the KPZ fixed point.

  TASEP is a microscopic (as opposed to continuum) model for (single lane) traffic (Burgers equation).

  With Mustazee Rahman (MIT) we are have results for eg. position of shock (typically the difference of two GOE's. Earlier special cases by Ferrari-Nejjar.)