# The stability of Kerr-de Sitter black holes

András Vasy (joint work with Peter Hintz)

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This talk is about the stability of Kerr-de Sitter (KdS) black holes, which are certain Lorentzian manifolds solving Einstein's equation – most of it will discuss what these black holes are and what stability means.

We adopt the convention is that Lorentzian metrics on an $n$-dimensional manifold have signature $(1, n-1)$. For instance, the Minkowski metric on $\mathbb{R}^4 = \mathbb{R}^{1+3}$, with coordinates $z_0, z_1, z_2, z_3$, is

$$g = dz_0^2 - dz_1^2 - dz_2^2 - dz_3^2.$$

Here $z_0$ is 'time', $(z_1, z_2, z_3)$ 'space', but there are many other timelike and spacelike coordinate functions on it! Here $f$ timelike means $g^{-1}(df, df) > 0$, spacelike means $g^{-1}(df, df) < 0$.

In 4 dimensions Einstein's equation in vacuum is an equation for the metric tensor of the form

$$\mathrm{Ric}(g) + \Lambda g = 0,$$

where $\Lambda$ is the cosmological constant, and $\mathrm{Ric}(g)$ is the Ricci curvature of the metric. If there were matter present, there would be a non-trivial right hand side of the equation, given by (a modification of) the matter's stress-energy tensor.

E.g. the Minkowski metric solves this with $\Lambda = 0$. Another solution, with $\Lambda > 0$, is de Sitter space, which is the one-sheeted hyperboloid in one higher dimensional Minkowski space.

We will be interested in $\Lambda > 0$; note that the observed accelerating expansion of the universe is consistent with a positive cosmological constant, which plays the role of a positive vacuum energy density, thus is of physical interest.

In local coordinates, the Ricci curvature is a non-linear expression in up to second derivatives of $g$; thus, this is a partial differential equation. The type of PDE that Einstein's equation is most similar to (with issues!) are (tensorial, non-linear) wave equations. The typical formulation of such a wave equation is that one specifies 'initial data' at a spacelike hypersurface, such as $z_0 = C$, $C$ constant, in Minkowski space. For linear wave equations $\Box u = f$ on spaces like $\mathbb{R}^{1+3}$, the solution $u$ for given data exists globally and is unique.

The analogue of the question how solutions of Einstein's equation behave is: if one has a solution $u_0$ of $\Box u = 0$, say $u_0 = 0$ with vanishing data, we ask how the solution $u$ changes when we slightly perturb data (to be still close to 0). For instance, does $u$ stay close to $u_0$ everywhere? Does perhaps even tend to $u_0$ as $z_0 \to \infty$? This is the question of stability of solutions.

Very few properties of $\mathrm{Ric}$ matter for our purposes.

First, $\mathrm{Ric}$ is diffeomorphism invariant, so if $\Psi$ is a diffeomorphism, and $g$ solves Einstein's equation, then so does $\Psi^* g$. This means that if there is one solution, there are many (even with same IC). In practice (duality) this means that it may not be easy to solve the equation at all!

This already indicates that Einstein's equation is not quite a wave equation, but it can be turned into one by imposing extra gauge conditions. This is implemented using the second key property, the 2nd Bianchi identity, $\delta_g G_g \mathrm{Ric}(g) = 0$ for all $g$, where $\delta_g$ is the symmetric gradient, and $G_g r = r - \frac{1}{2}(\mathrm{tr}_g\, r) g$. An implementation is

$$\mathrm{Ric}(g) + \Lambda g - \Phi(g, t) = 0,$$

where

$$\Phi(g, t) = \delta_g^* \Upsilon(g), \ \Upsilon(g) = g g_0^{-1} \delta_g G_g g_0.$$

This enabled Choquet-Bruhat to show local well-posedness: $\Upsilon = 0$.

The first stability results were obtained for Minkowski space and de Sitter space, respectively, and are due to Christodoulou and Klainerman (1990s), later simplified by Lindblad and Rodnianski (2000s) (and extended by Bieri and Zipser, and in a different direction by Hintz and V.), resp. Friedrich (1980s).

Our result with Peter Hintz is the stability of Kerr-de Sitter (KdS) black holes (slowly rotating). These are family of metrics depending on two parameters, called mass $m$ and angular momentum $a$. The $a = 0$ members of the family are called the Schwarzschild-de Sitter (SdS) black holes; $h_{\mathbb{S}^2}$ the metric on $\mathbb{S}^2$:

$$g = \mu(r)\, dt^2 - \mu(r)^{-1}\, dr^2 - r^2\, h_{\mathbb{S}^2},\ \ \mu(r) = 1 - \frac{2m}{r} - \frac{\Lambda r^2}{3},$$

- $\Lambda = 0$ gives the Schwarzschild metric, discovered in about a month after Einstein's 1915 paper.
- $m = 0$ gives the de Sitter metric.

Recall:

$$g = \mu(r)\,dt^2 - \mu(r)^{-1}\,dr^2 - r^2\,h,\ \ \mu(r) = 1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}.$$

- $\mu(r) = 0$ has two positive solutions $r_+, r_-$ if $m, \Lambda > 0$; if $\Lambda = 0$ the only root is $2m$, if $m = 0$, the only root is $\sqrt{3/\Lambda}$.
- In this form the metric makes sense where $\mu > 0$: $\mathbb{R}_t \times (r_-, r_+)_r \times \mathbb{S}^2$.
- However, $r = r_\pm$ are coordinate singularities; with $c_\pm$ smooth,

$$t_* = t - F(r),\ \ F'(r) = \pm(\mu(r)^{-1} + c_\pm(r)) \text{ near } r = r_\pm$$

  desingularizes them and extends the metric to

$$\mathbb{R}_{t_*} \times (0, \infty)_r \times \mathbb{S}^2_\omega,$$

- $r = r_-$ is called the *event horizon*, $r = r_+$ the *cosmological horizon*; they are very similar for the geometry.
- $\partial_t$ is a Killing vector field, i.e. translation in $t$ preserves the metric, and it is spherically symmetric.

Without specifying the general KdS metric, we just mention that the underlying manifold is a $\mathbb{R}_{t_*} \times (0, \infty)_r \times \mathbb{S}^2$, and $\partial_{t_*}$ is a Killing vector field, i.e. translation in $t_*$ preserves the metric. These metrics are axisymmetric around the axis of rotation.

In general, for a manifold $M$ with $\Sigma_0$ a codimension 1 hypersurface, the initial data are a Riemannian metric $h$ and a symmetric 2-cotensor $k$ which satisfy the constraint equations (needed for solvability), and one calls a Lorentzian metric $g$ on $M$ a solution of Einstein's equation with initial data $(\Sigma_0, h, k)$ if the pull-back of $g$ to $\Sigma_0$ is $-h$, and $k$ is the second fundamental form of $\Sigma_0$ in $M$.

Our main result is the global non-linear asymptotic stability of the Kerr-de Sitter family for the initial value problem for small angular momentum $a$ on the space

$$\Omega = [0, \infty)_{t_*} \times [r_- - \delta, r_+ + \delta]_r \times \mathbb{S}^2.$$

**Theorem (Hintz-V '16; informal version)**

*Suppose $(h, k)$ are smooth initial data on $\Sigma_0$, satisfying the constraint equations, which are close to the data $(h_{b_0}, k_{b_0})$ of a Schwarzschild–de Sitter spacetime, $b_0 = (m_0, 0)$, in a high regularity norm. Then there exist a solution $g$ of Einstein's equation in $\Omega$ attaining these initial data at $\Sigma_0$, and black hole parameters $b = (m, a)$ which are close to $b_0$, so that*

$$g - g_b = \mathcal{O}(e^{-\alpha t_*})$$

*for a constant $\alpha > 0$ independent of the initial data; that is, $g$ decays exponentially fast to the Kerr–de Sitter metric $g_b$. Moreover, $g$ and $b$ are quantitatively controlled by $(h, k)$.*

What the theorem states is that the metric 'settles down to' a Kerr-de Sitter metric at an exponential rate while emitting energy through gravitational waves (recently detected by LIGO). Note that even if we perturb a Schwarzschild-dS metric, we get a KdS limit!

Strongest $\Lambda = 0$ black hole result is the stability of linearized Schwarzschild, plus Teukolsky (very recently announced): Dafermos-Holzegel-Rodnianski (2016), and a restricted (symmetry) stability of Schwarzschild (Klainerman-Szeftel, 2017). There has been extensive research in the area, including works by (in addition to the authors already mentioned) Wald, Kay, Finster, Kamran, Smoller, Yau, Tataru, Tohaneanu, Marzuola, Metcalfe, Sterbenz, Andersson, Blue, Donninger, Schlag, Soffer, Sá Barreto, Wunsch, Zworski, Bony, Häfner, Dyatlov, Luk, Ionescu, Shlapentokh-Rothman...

The analytic framework we use

- non-elliptic linear global analysis with coefficients of finite Sobolev regularity,
- with a simple global Nash-Moser iteration to deal with the loss of derivatives corresponding to both non-ellipticity and trapping

gives global solvability for quasilinear wave equations like the gauged Einstein's equation provided

- certain dynamical assumptions are satisfied (only trapping is normally hyperbolic trapping, with an appropriate subprincipal symbol condition) and
- there are no exponentially growing modes (with a precise condition on non-decaying ones), i.e. non-trivial solutions of the linearized equation at $g_{b_0}$ of the form $e^{-i\sigma t_*}$ times a function of the spatial variables $r, \omega$ only, with $\operatorname{Im} \sigma > 0$.

Unfortunately, in the harmonic/wave/DeTurck gauge, while the dynamical assumptions are satisfied, there *are* growing modes, although only a finite dimensional space of these. The key to proving the theorem (given the analytic background) is to overcome this issue.

The Kerr-de Sitter family automatically gives rise to non-decaying modes with $\sigma = 0$, but as these correspond to non-linear solutions, one may expect these not to be a problem with some work.

One might then expect that the other non-decaying (including growing!) modes come from the diffeomorphism invariance, i.e. gauge issues, but this is not true at this stage!

However, we can arrange for a partial success: we can modify $\Phi$ by changing $\delta_g^*$ by a 0th order term:

$$\widetilde{\delta}^* \omega = \delta^*_{g_0} \omega + \gamma_1 \, dt_* \otimes_s \omega - \gamma_2 g_0 \, \mathrm{tr}_{g_0}(dt_* \otimes_s \omega),$$

$$\Phi(g, t) = \widetilde{\delta}^* \Upsilon(g).$$

For suitable choices of $\gamma_1, \gamma_2 \gg 0$, this preserves the dynamical requirements, and while the gauged Einstein's equation does still have growing modes, it has a new feature:

$$\widetilde{\Box}^{\mathrm{CP}}_g = 2\delta_g G_g \widetilde{\delta}^*, \ \ g = g_{b_0}$$

has no non-decaying modes! (There was no reason to expect that the DeTurck gauge is well-behaved in any way except in a high differential order sense, relevant for the local theory!)

Such a change to the gauge term, called 'constraint damping', has been successfully used in the numerical relativity literature by Pretorius and others, following the work of Gundlach et al, to damp numerical errors in $\Upsilon(g) = 0$; here we show rigorously why such choices work well.

This change is useful because it means that, for $g = g_{b_0}$, any non-decaying mode $r$ of the linearized gauged Einstein equation is a solution $D_g(\mathrm{Ric}(g) + \Lambda g)r = 0$, and thus is geometric, given by infinitesimal diffeomorphisms or the KdS family (Ishibashi, Kodama and Seto).

Concretely then, ignoring the KdS family induced zero modes, we take

$$\Phi(g, t; \theta) = \widetilde{\delta}^*(\Upsilon(g) - \theta),$$

where $\theta$ is an unknown, lying in a finite dimensional space $\Theta$ of gauge terms of the form $D_{g_{b_0}}\Upsilon(\delta^*_{g_{b_0}}(\chi\omega))$, where $\chi \equiv 1$ for $t_* \gg 1$, $\chi \equiv 0$ near $t_* = 0$, and such that $\delta^*_{g_{b_0}}\omega$ is a non-decaying resonance of the gauged Einstein operator, and solve

$$\mathrm{Ric}(g) + \Lambda g - \Phi(g, t; \theta) = 0$$

for $g$ and $\theta$, with $g - g_{b_0}$ in a decaying function space.