From quantum integrability to Schubert calculus

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Australian Government
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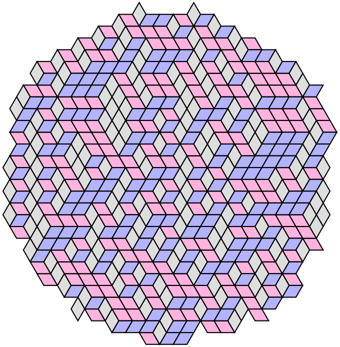
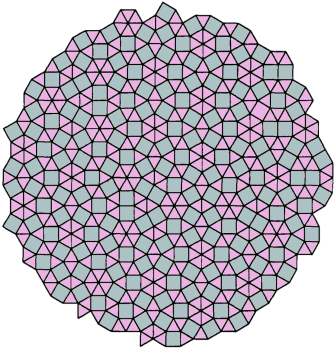
Introduction

These two random tiling models:





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- They are related to Schubert calculus.

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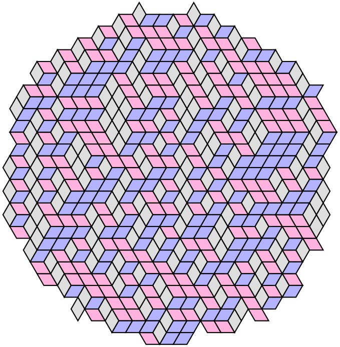

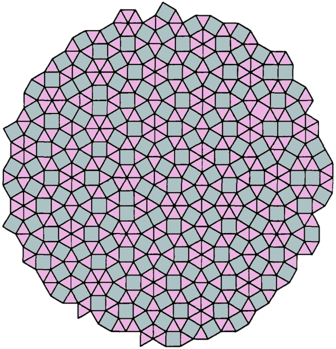

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Exactly solvable 2d lattice models → Symmetric polynomials

- Symmetric polynomials[1] appear in many areas of pure mathematics (combinatorics, representation theory, etc), as well as in applied mathematics and mathematical physics (random matrix theory, integrable systems, etc).
- In many cases, there is an underlying "integrability": certain families of symmetric polynomials can be described explicitly in terms of two-dimensional exactly solvable lattice models.
- Sometimes, this integrability can be extended to the computation of structure constants of the ring of symmetric polynomials in that particular basis (e.g., Schur functions and Littlewood–Richardson coefficients).
- There are deep connections to (enumerative, algebraic) geometry, in particular to Schubert calculus.

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Schur polynomials: motivation

Schur polynomials are the most famous family of symmetric polynomials. They are homogeneous polynomials with integer coefficients.

- They form a basis of the ring of symmetric polynomials (i.e., a basis of $\mathbb{Z}[x_1, \ldots, x_n]^{\mathcal{S}_n}$ as a graded $\mathbb{Z}$-module for each n).
- They are the characters of polynomial irreducible representations of the general linear group GL_n.
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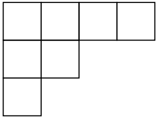
Schur polynomials: definition

To a Young diagram λ (or its associated Maya diagram), one associates the Schur polynomial $s^\lambda(x_1, \ldots, x_n)$ which is a sum over lozenge tilings:



where each light pink lozenge at row i contributes a weight x_i.
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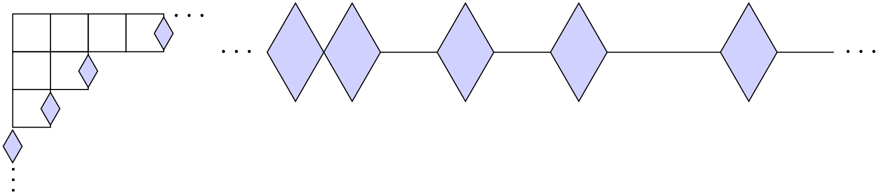

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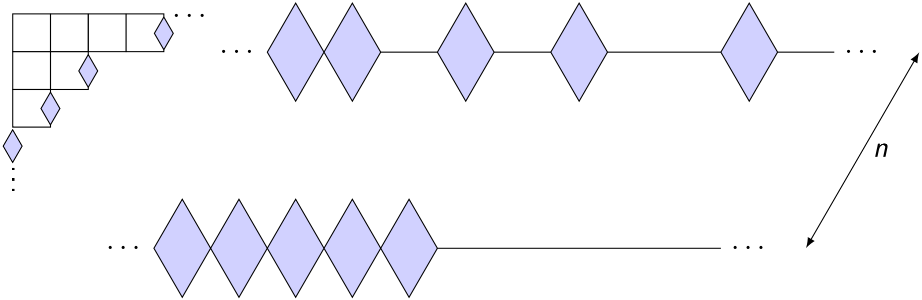

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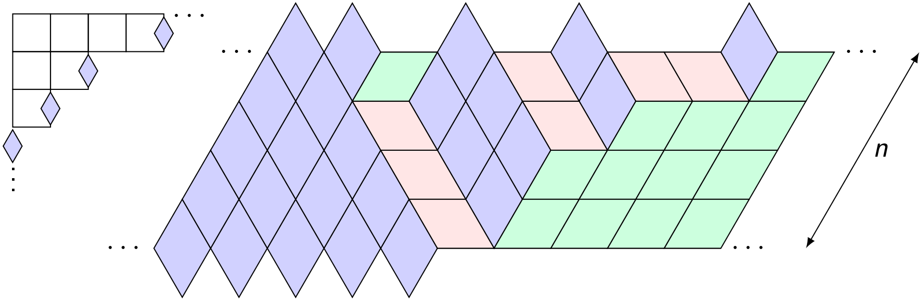

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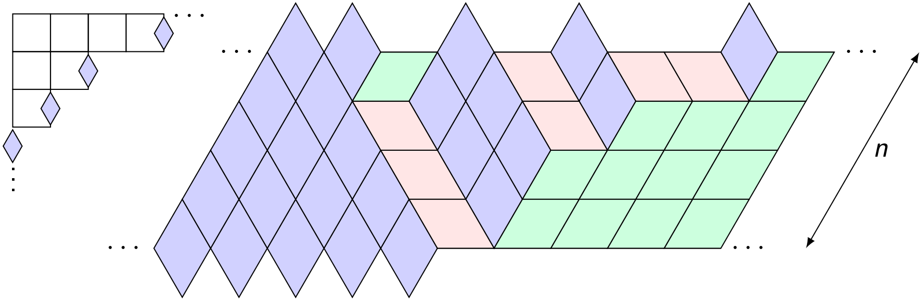

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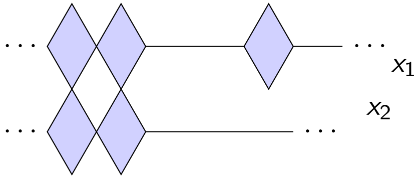

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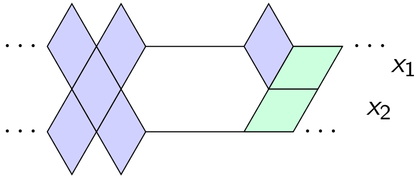

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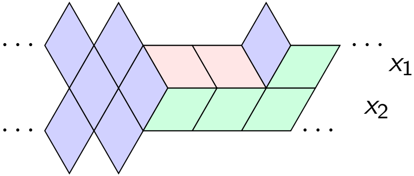
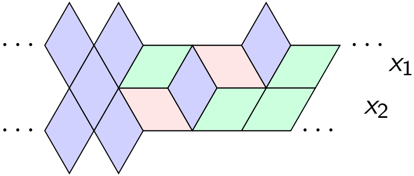
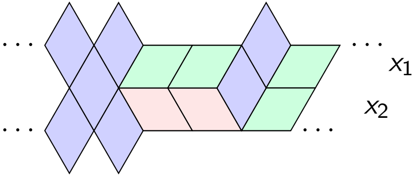
$$s_{\square\square}(x_1, x_2) = x_1^2$$



$$+\, x_1 x_2$$



$$+\, x_2^2$$



Lozenge tilings as an exactly solvable model

- The lozenge tiling model can be reformulated as a vertex model: the rational 5-vertex model.
- This model is quantum integrable; it is based on the algebra $\mathfrak{sl}_2$ in the spin 1/2 irrep.
- Actually, it is equivalent to Non-Intersecting Lattice Paths and therefore free fermionic / determinantal.
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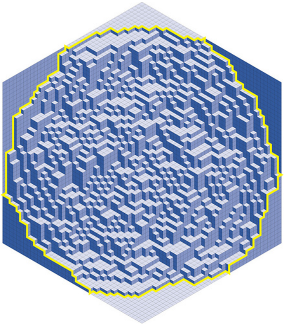

The Littlewood–Richardson problem

- Whenever one has a basis of a ring (such as the s^λ for the ring of symmetric polynomials), one can ask about structure constants:

$$s^\lambda s^\mu = \sum_\nu c_\nu^{\lambda,\mu} s^\nu$$

- In the case of Schur polynomials, there is a representation-theoretic interpretation (decomposition of tensor product of irreducible representations of the general or special linear group).
- In the case of Schur (or Schubert/Grothendieck) polynomials, there is a geometric interpretation (intersection theory on the Grassmannian).

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Example

$$\left(s^{\square}\right)^2 = s^{\square\square} + s^{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$$

The Littlewood–Richardson rule

- We are looking for a manifestly positive formula for $c_{\nu}^{\lambda,\mu}$.
- Such a formula was first proposed by Littlewood and Richardson in 1934 in terms of tableaux, and proved by Schützenberger in 1977.
- Another rule was given by Knutson and Tao (2003) in their proof of the saturation conjecture: puzzles.
  It is the form that most explicitly displays the underlying quantum integrability!
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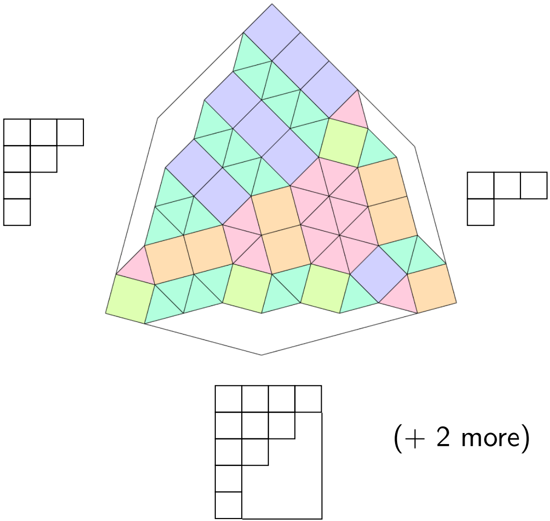

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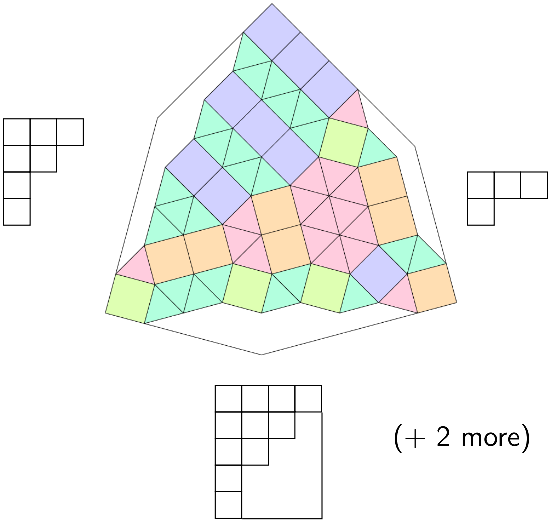

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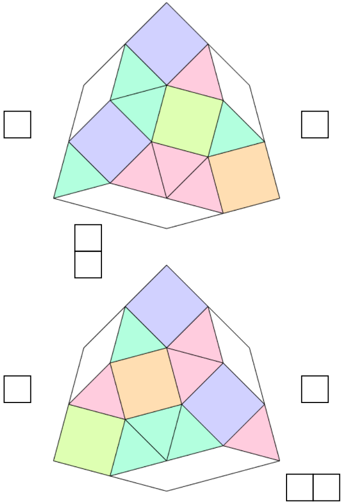

The square-triangle tiling: history

- 1993: M. Widom introduces the square-triangle model (in relation to quasi-crystals), deforms it into a regular triangular lattice ($\sim$ puzzles) and proves integrability.
- 1994: P. Kalugin (partially) solves the Coordinate Bethe Ansatz equations (size$\to\infty$).
- 1997–2006: J. de Gier and B. Nienhuis reinvestigate it, noticing that it's a singular limit of an $\mathfrak{sl}_3$ model.
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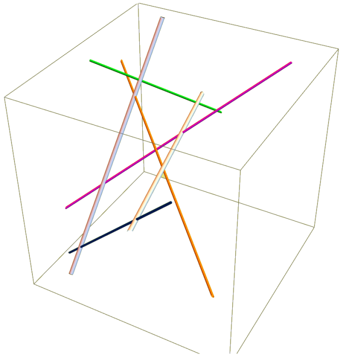

Cohomology theories and QIS

- These questions reduce to calculations in the cohomology ring of the space of configurations, e.g. Grassmannians.
- The recently discovered connection between QIS and cohomology theories (Okounkov et al; see also Knutson+ZJ, Rimanyi+Tarasov+Varchenko), itself motivated by relations to SUSY gauge theory (Nekrasov+Shatashvili), allows in particular to express appropriate cohomology classes (e.g., Schur polynomials) as partition functions of QIS.
- The integrability of the product rule is an extra ingredient, whose geometric meaning was recently uncovered by Knutson+ZJ.
- More generally, we expect to be able to express structure constants of the cohomology of Nakajima quiver varieties (and beyond).

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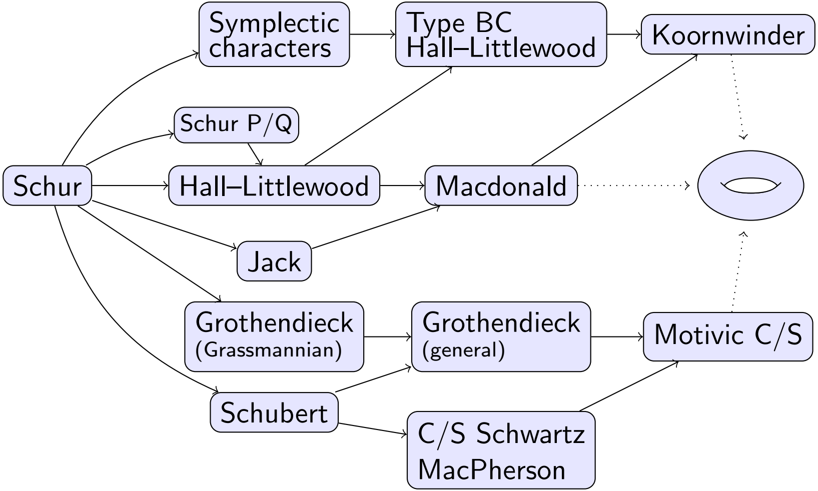

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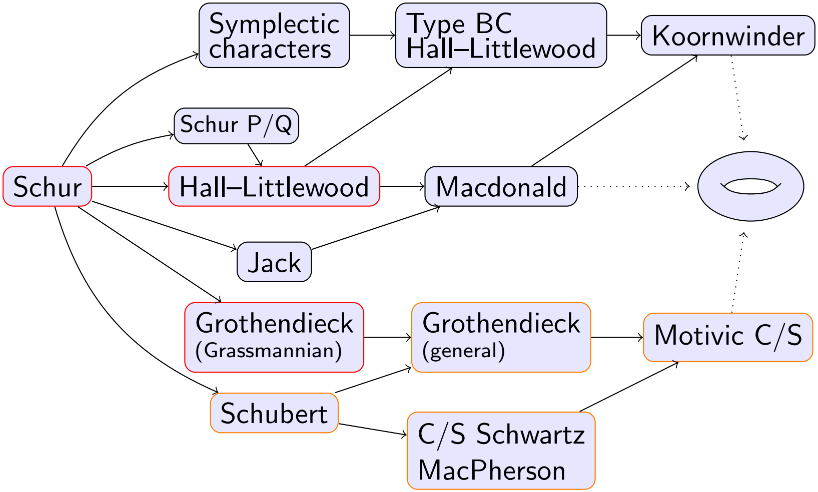

Case of Grothendieck polynomials

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- They are a one-parameter deformation of Schur polynomials.
- The corresponding integrable model is implicit in the work of Fomin and Kirillov (1994).
- Here we first consider the case of the Grassmannian (analogue of Schur polynomials, rather than general Schubert polynomials).
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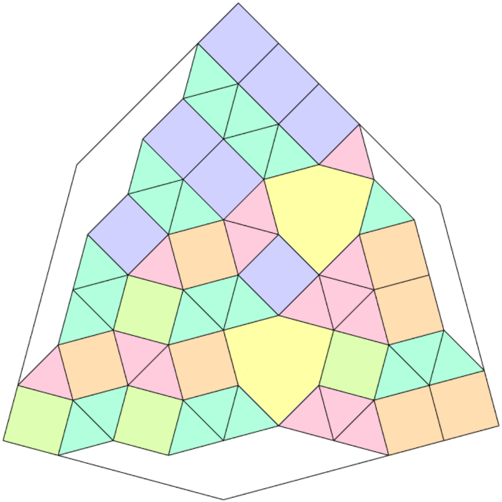
The square-triangle shield model



Case of Hall–Littlewood polynomials

- Hall–Littlewood polynomials $P_\lambda(t; x_1, \ldots, x_n)$ are a family of polynomials depending on one parameter t which interpolate between two bases: Schur polynomials ($t = 0$) and symmetrized monomials ($t = 1$).
- Remarkably, to express them as partition functions of a lattice model requires a trigonometric $\mathfrak{sl}_2$ model with infinite spin, where t plays the role of quantum parameter.
- The integrable model for their product rule is a $\mathfrak{sl}_3$ infinite spin (parabolic Verma module) model, best expressed in terms of honeycombs. [ZJ '18]

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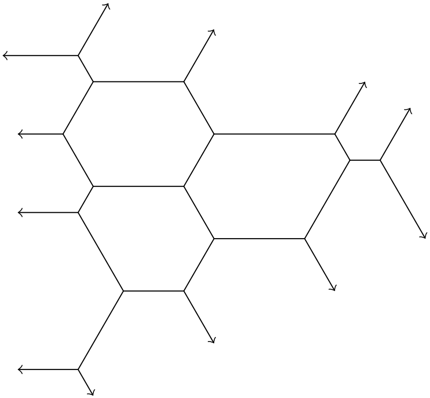
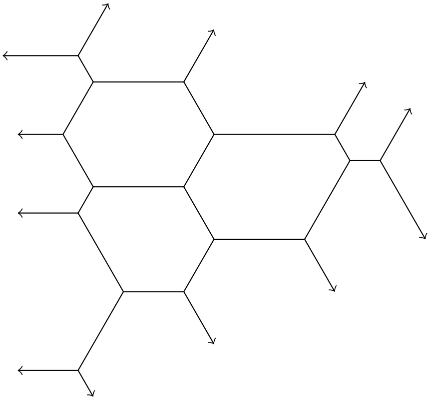
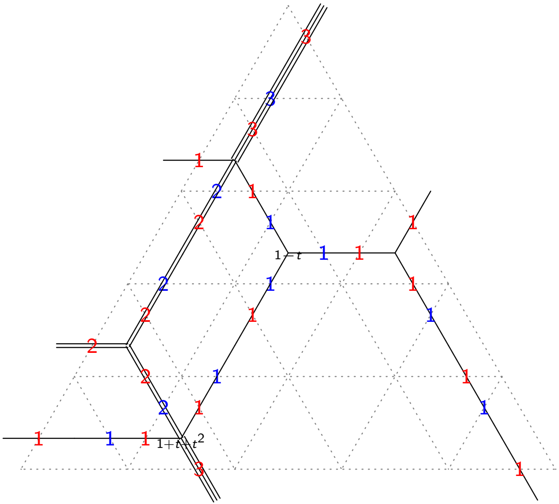
Honeycombs



Honeycombs





Beyond $\mathfrak{sl}_2$

- All the families of symmetric polynomials considered so far are based on the algebra $\mathfrak{sl}_2$, i.e., A_1.
- The product rules turned out to be *also* given by integrable models, but based on the algebra $\mathfrak{sl}_3$, i.e., A_2.
- In order to proceed further, one needs to extend this construction to other Lie algebras. In particular, Schubert calculus in d-step flag varieties (i.e., Schubert or general Grothendieck polynomials) are related to A_d.
- We have achieved this (partially) in our recent paper arXiv:1706.10019 with A. Knutson, thus providing a partial answer to the venerable 19^{th} century problem of Schubert calculus.

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QIS and root systems

Family of polynomials ↔ model ❶, their product rule ↔ model ❷.

	model ❶	dim rep ❶	model ❷	dim rep ❷
$d = 1$	A_1	2		
$d = 2$	A_2	3		
$d = 3$	A_3	4		
$d = 4$	A_4	5		
$d \geq 5$	A_d	$d + 1$		

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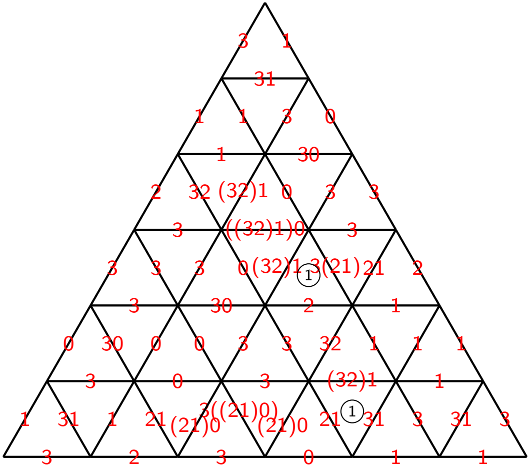
$d = 1$	A_1	2	A_2	
$d = 2$	A_2	3	D_4	
$d = 3$	A_3	4	E_6	
$d = 4$	A_4	5	E_8	
$d \geq 5$	A_d	$d + 1$	Kac–Moody?	

QIS and root systems

Family of polynomials $\leftrightarrow$ model ❶, their product rule $\leftrightarrow$ model ❷.

	model ❶	dim rep ❶	model ❷	dim rep ❷
$d=1$	A_1	2	A_2	3
$d=2$	A_2	3	D_4	8
$d=3$	A_3	4	E_6	27
$d=4$	A_4	5	E_8	$248+1$
$d\geq 5$	A_d	$d+1$	Kac–Moody?	∞

$d = 3$ example


3 1
31
1 1 3 0
1 30
2 32 (32)1 0 3 3
3 ((32)1)0 3
3 3 3 0 (32)1 1 3(21) 21 2
1
3 30 2 1
0 30 0 0 3 3 32 1 1 1
3 0 3 (32)1 1
1 31 1 21 (21)0 3((21)0) (21)0 21 1 31 3 31 3
3 2 3 0 1 1