Bogoliubov theory in the Gross-Pitaevskii regime

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joint work with
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The dilute Bose gas in 3d

N bosons enclosed in a cubic box Λ of side length L, described by

$$H_N = -\sum_{j=1}^{N} \Delta_{x_j} + \sum_{1 \le i < j \le N} V(x_i - x_j)\,, \qquad \rho a^3 \ll 1$$

Long-standing goals: for $N, L \to \infty$ and $\rho = N/|\Lambda|$ fixed

- **prove the occurrence of condensation**
  - hard-core bosons at half filling [Dyson-Lieb-Simon,'78]
  - renormalization group results:
    [Benfatto '94], [Balaban-Feldman-Knörrer-Trubowitz '08-'16]
- **compute thermodynamic functions**
  - ground state energy: [Dyson'57], [Lieb-Yngvason,'98], [Erdös-Schlein-Yau,'08], [Giuliani-Seiringer '09], [Yau-Yin, '13], [Brietzke-Solovej '17]
- **low lying excitation spectrum**

Bogoliubov theory

-) Fock space Hamiltonian, momentum space

$$H = \sum_{p\in\Lambda^*} p^2 a_p^* a_p + \frac{1}{2|\Lambda|} \sum_{p,q,r\in\Lambda^*} \widehat{V}(r) a_{p+r}^* a_q^* a_p a_{q+r} \,, \qquad \Lambda^* = \frac{2\pi}{L}\mathbb{Z}^3$$

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-) Since he expected BEC in low energy states replaced a_0, a_0^* by $N^{1/2}$

$$H = \frac{N(N-1)}{2|\Lambda|}\widehat{V}(0) + \sum_{p\in\Lambda_+^*} \left[|p|^2 + \frac{N}{|\Lambda|}\widehat{V}(p)\right] a_p^* a_p \qquad \Lambda_+^* = \frac{2\pi}{L}\mathbb{Z}^3 \setminus \{0\}$$

$$+ \frac{N}{2|\Lambda|} \sum_{p\in\Lambda_+^*} \widehat{V}(p)\left(a_p a_{-p} + a_p^* a_{-p}^*\right) + (\text{cubic}) + (\text{quartic})\,, \qquad N/|\Lambda| := \rho$$

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-) Neglecting cubic and quartic contributions, diagonalization leads to

$$H_B = E_{N,\Lambda} + \sum_{p\in\Lambda_+^*}\sqrt{p^4 + 2\rho p^2 \widehat{V}(p)}\, n_p \qquad n_p \in \mathbb{N}$$

$$E_{N,\Lambda} = \frac{N}{2}\rho\widehat{V}(0) - \frac{1}{4}\sum_{p\in\Lambda_+^*}\frac{(\rho\widehat{V}(p))^2}{p^2} - \frac{1}{2}\sum_{p\in\Lambda_+^*}\left[p^2 + \rho\widehat{V}(p) - \sqrt{p^4 + 2\rho\widehat{V}(p)p^2} - \frac{1}{4}\sum_{p\in\Lambda_+^*}\frac{(\rho\widehat{V}(p))^2}{p^2}\right]$$

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-) Thermodinamic limit and emergence of the scattering length

$$e_0 = \lim_{N,|\Lambda|\to\infty,\ \rho=N/|\Lambda|}\frac{E_{N,\Lambda}}{N} = 4\pi\rho(\mathfrak{a}_0 + \mathfrak{a}_1) + 4\pi\rho\mathfrak{a}_0\frac{128}{15\pi}\sqrt{\rho\mathfrak{a}_0^3} + o(\rho^{3/2}\mathfrak{a}_0^{5/2})$$

where $\mathfrak{a}_0 = (8\pi)^{-1}\widehat{V}(0)$ and $\mathfrak{a}_1 = \mathcal{O}(\mathfrak{a}_0^2/R)$, with R the range of the interaction.

Bogoliubov theory and rigorous results

Expectation for the ground state energy for particle:

$$e_0 = 4\pi\rho(\mathfrak{a}_0 + \mathfrak{a}_1) + 4\pi\rho\mathfrak{a}_0 \frac{128}{15\pi}\sqrt{\rho\mathfrak{a}_0^3} + o(\rho^{3/2}\mathfrak{a}_0^{5/2})$$

$$\overset{?}{=} 4\pi\rho\mathfrak{a} + 4\pi\rho\mathfrak{a}\frac{128}{15\pi}\sqrt{\rho\mathfrak{a}^3} + o(\rho^{3/2}\mathfrak{a}^{5/2})$$

Recall: $8\pi\mathfrak{a} = \int V(x)f(x)dx$ with f solution of $(-\Delta + \frac{1}{2}V)f = 0$ with $f(x) \to 1$ as $|x| \to \infty$. For $\mathfrak{a}_0 = (8\pi)^{-1}\widehat{V}(0) \ll R$ we may write

$$\mathfrak{a} = \mathfrak{a}_0 + \mathfrak{a}_1 + \mathfrak{a}_2 + \dots \qquad \text{with} \qquad \mathfrak{a}_j = \mathfrak{a}_0\,(\mathfrak{a}_0/R)^j$$

Rigorous results

- Leading order: [Dyson'57], [Lieb-Yngvason,'98]
- Second order: upper and lower bounds for regimes s.t. $\mathfrak{a}_1 \gg \mathfrak{a}_0\sqrt{\rho\mathfrak{a}_0^3} \gg \mathfrak{a}_2$ [Lieb-Solovej, '01 & '04], [Giuliani-Seiringer '09], [Brietzke-Solovej '17];
- Second order for $\rho\mathfrak{a}^3 \ll 1$, only upper bounds available [Erdös-Schlein-Yau,'08], [Yau-Yin, '13]

Bogoliubov theory and rigorous results

Bogoliubov approximation has been proved to be valid for bosons in the mean field regime:

$$H_N^{mf} = -\sum_{j=1}^{N} \Delta_{x_j} + \frac{1}{N} \sum_{1\le i<j\le N} V(x_i - x_j), \qquad |\Lambda| = 1$$

where $\mathfrak{a}_0, \mathfrak{a}_0/R \sim N^{-1}$, hence $\mathfrak{a}_j \sim N^{-(j+1)}$ and $\mathfrak{a}_1 \gg \mathfrak{a}_0\sqrt{\rho\mathfrak{a}_0^3} \gg \mathfrak{a}_2$.

Results for the homogeneous case [Seiringer '11]:

- Condensation with rate of convergence: $1 - \langle \varphi_0, \gamma_N^{(1)} \varphi_0 \rangle \le CN^{-1}$
- Ground state energy at second order

$$E_N^{mf} = \frac{(N-1)\widehat{V}(0)}{2} - \frac{1}{2}\sum_{p\in\Lambda_+^*} \left[p^2 + \kappa\widehat{V}(p) - \sqrt{|p|^4 + 2\kappa|p|^2\widehat{V}(p)} \right] + o(1)$$

- Bogoliubov spectrum of elementary excitations

$$\sum_{p\in\Lambda_+^*} n_p \sqrt{|p|^4 + 2\kappa p^2 \widehat{V}(p)} + o(1), \quad n_p \in \mathbb{N}$$

Further results: [Grech-Seiringer '13],[Lewin-Nam-Serfaty-Solovey '14], [Derezinski-Napiorkovski '14], [Pizzo '15]

Bogoliubov theory in the Gross-Pitaevskii regime

Consider N bosons in a cubic box Λ described by

$$H_N = -\sum_{i=1}^{N} \Delta_{x_i} + \kappa \sum_{i<j}^{N} N^2 V\big(N(x_i - x_j)\big)\,, \qquad |\Lambda| = 1$$

- If $\kappa V(x)$ has scattering length $\mathfrak{a}$, then $\kappa N^2 V(Nx)$ has scattering length $\mathfrak{a}/N \longrightarrow$ dilute regime $\rho a^3 = N^{-2}$
- since $\mathfrak{a}_0/R = \mathcal{O}(\kappa)$ all terms in the Born series of the scattering length contribute to the same order in N: we cannot replace first and second Born approximation with the full scattering length!

Relevance:

- physically relevant for the description of strong and short range interactions among atoms in BEC experiments
- mathematically challenging since correlations among the particles play a crucial role to understand statical and dynamical properties of the system
- H_N equivalent to the Hamiltonian for N bosons in a box with $L = N$ interacting through a fixed potential κV, *i.e.* $\rho = N/L^3 = N^{-2}$

Bogoliubov theory in the Gross-Pitaevskii regime

N bosons in $\Lambda = [0;1]^{\times 3}$, periodic boundary conditions

$$H_N = \sum_{p \in \Lambda^*} p^2 a_p^* a_p + \frac{\kappa}{2N} \sum_{p,q,r \in \Lambda^*} \widehat{V}(r/N)\, a_{p+r}^* a_{q-r}^* a_p a_q \,, \qquad \Lambda_* = 2\pi\mathbb{Z}^3$$

From [Lieb-Seiringer-Yngvason, '00] the ground state energy of H_N at leading order is

$$E_N = 4\pi \mathfrak{a}\, N + o(N)$$

From [Lieb-Seiringer, '02] the one particle reduced density $\gamma_N^{(1)}$ associated to the ground state of H_N is such that in trace norm

$$\gamma_N^{(1)} \xrightarrow[N\to\infty]{} |\varphi_0\rangle\langle\varphi_0|$$

where $\varphi_0(x) = 1$ for all $x \in \Lambda$.

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$$E_N = 4\pi \mathfrak{a}\, N + o(N) \qquad 8\pi\mathfrak{a} = \kappa \int \mathrm{d}x\, f(x) V(x)$$

Note that $\langle \varphi_0^{\otimes N} H_N \varphi_0^{\otimes N} \rangle = \frac{(N-1)\kappa \widehat{V}(0)}{2} \gg 4\pi\mathfrak{a}N$

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[Boccato-Brennecke-C.-Schlein '18] For $\kappa > 0$ small enough, the ground state energy of H_N is

$$E_N = 4\pi(N-1)\mathfrak{a}_N - \frac{1}{2}\sum_{p \in \Lambda_+^*} \left[p^2 + 8\pi\mathfrak{a} - \sqrt{|p|^4 + 16\pi\mathfrak{a}p^2} - \frac{(8\pi\mathfrak{a})^2}{2p^2} \right] + \mathcal{O}(N^{-1/4})$$

where $\Lambda_+^* = 2\pi\mathbb{Z}^3 \setminus \{0\}$ and

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where $\Lambda_+^* = 2\pi\mathbb{Z}^3 \setminus \{0\}$ and

$$8\pi\mathfrak{a}_N = \kappa \widehat{V}(0) - \frac{1}{2N} \sum_{p_1 \in \Lambda_+^*} \frac{\kappa^2 \widehat{V}^2(p_1/N)}{2p_1^2}$$
$$+ \sum_{m=2}^{+\infty} \frac{(-1)^m \kappa^m}{(2N)^m} \sum_{p_1,\dots,p_m \in \Lambda_+^*} \frac{\widehat{V}(p_1/N)}{p_1^2} \left[\prod_{j=1}^{m-1} \frac{\widehat{V}((p_j - p_{j+1})/N)}{p_{j+1}^2} \right] \widehat{V}(p_m/N)$$

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where $\Lambda_+^* = 2\pi\mathbb{Z}^3 \setminus \{0\}$ and

$$\left|4\pi(N-1)\mathfrak{a}_N - 4\pi(N-1)\mathfrak{a}\right| \leq C$$

$$8\pi\mathfrak{a}_N = \kappa\widehat{V}(0) - \frac{1}{2N}\sum_{p_1\in\Lambda_+^*}\frac{\kappa^2\widehat{V}^2(p_1/N)}{2p_1^2}$$

$$+ \sum_{m=2}^{+\infty}\frac{(-1)^m\kappa^m}{(2N)^m}\sum_{p_1,\dots,p_m\in\Lambda_+^*}\frac{\widehat{V}(p_1/N)}{p_1^2}\left[\prod_{j=1}^{m-1}\frac{\widehat{V}((p_j-p_{j+1})/N)}{p_{j+1}^2}\right]\widehat{V}(p_m/N)$$

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The spectrum of $H_N - E_N$ below an energy ζ consists of

$$\sum_{p\in\Lambda_+^*} n_p\sqrt{|p|^4 + 16\pi\mathfrak{a}|p|^2} + \mathcal{O}(N^{-1/4}(1+\zeta^3))$$

with $n_p \in \mathbb{N}$ and $n_p \neq 0$ for finitely many $p \in \Lambda_+^*$ only.

Step 1: removing particles in the Bose-Einstein condensate

For $\psi_N \in L^2_s(\Lambda^N)$ and $\varphi_0 \in L^2(\Lambda)$ [Lewin-Nam-Serfaty-Solovej '12]

$$\psi_N = \alpha_0\, \varphi_0^{\otimes N} + \alpha_1 \otimes_s \varphi_0^{\otimes N-1} + \ldots + \alpha_j \otimes_s \varphi_0^{\otimes N-j} + \ldots + \alpha_N\,,$$

where $\alpha_j \in L^2(\Lambda)^{\otimes_s j}$ and $\alpha_j \perp \varphi_0$.

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Unitary map: $U_N(\varphi_0) : L^2_s(\Lambda^N) \longrightarrow \mathcal{F}^{\leq N}_{\perp\varphi_0} = \bigoplus_{n=0}^{N} L^2_{\perp\varphi_0}(\Lambda)^{\otimes_s n}$

$$\psi_N \longrightarrow U_N(\varphi_0)\psi_N = \{\alpha_0,\, \alpha_1,\, \ldots, \alpha_N, 0, 0, \ldots\}$$

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Conjugation with U_N reminds of Bogoliubov approximation

$$U_N\, a_0^* a_0\, U_N^* = N - \mathcal{N}_+ \qquad U_N\, a_0^* a_p\, U_N^* = \sqrt{N - \mathcal{N}_+}\, a_p$$

$$U_N\, a_p^* a_q\, U_N^* = a_p^* a_q \qquad U_N\, a_p^* a_0\, U_N^* = a_p^* \sqrt{N - \mathcal{N}_+}$$

$$\mathcal{N}_+ = \sum_{p \in \Lambda^* \setminus \{0\}} a_p^* a_p$$

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Excitation Hamiltonian: $\mathcal{L}_N = U_N H_N U_N^* : \mathcal{F}_+^{\leq N} \to \mathcal{F}_+^{\leq N}$

$$\begin{aligned}\mathcal{L}_N = &\frac{N-1}{2N}\kappa\widehat{V}(0)(N-\mathcal{N}_+) + \frac{\kappa\widehat{V}(0)}{2N}\mathcal{N}_+(N-\mathcal{N}_+)\\ &+\sum_{p\in\Lambda_+^*} p^2 a_p^* a_p + \sum_{p\in\Lambda_+^*}\kappa\widehat{V}(p/N)\, a_p^* \left(\frac{N-1-\mathcal{N}_+}{N}\right) a_p\\ &+\frac{\kappa}{2}\sum_{p\in\Lambda_+^*}\widehat{V}(p/N)\left[a_p^* \frac{(N-\mathcal{N}_+)(N-1-\mathcal{N}_+)}{N^2} a_{-p}^* + \text{h.c.}\right]\\ &+\frac{\kappa}{\sqrt{N}}\sum_{p,q\in\Lambda_+^*:p+q\neq 0}\widehat{V}(p/N)\left[b_{p+q}^* a_{-p}^* a_q + a_q^* a_{-p} b_{p+q}\right]\\ &+\frac{\kappa}{2N}\sum_{p,q\in\Lambda_+^*,r\in\Lambda^*:r\neq -p,-q}\widehat{V}(r/N) a_{p+r}^* a_q^* a_p a_{q+r}\end{aligned}$$

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Excitation Hamiltonian: $\mathcal{L}_N = U_N H_N U_N^* : \mathcal{F}_+^{\leq N} \to \mathcal{F}_+^{\leq N}$

$$\begin{aligned}\mathcal{L}_N = & \frac{N-1}{2N}\kappa\widehat{V}(0)(N-\mathcal{N}_+) + \frac{\kappa\widehat{V}(0)}{2N}\mathcal{N}_+(N-\mathcal{N}_+) \\ & + \sum_{p\in\Lambda_+^*} p^2 a_p^* a_p + \sum_{p\in\Lambda_+^*} \kappa\widehat{V}(p/N)\, a_p^* \left(\frac{N-1-\mathcal{N}_+}{N}\right) a_p \\ & + \frac{\kappa}{2}\sum_{p\in\Lambda_+^*} \widehat{V}(p/N)\left[a_p^* \frac{(N-\mathcal{N}_+)(N-1-\mathcal{N}_+)}{N^2} a_{-p}^* + \text{h.c.}\right] \\ & + \frac{\kappa}{\sqrt{N}} \sum_{p,q\in\Lambda_+^*: p+q\neq 0} \widehat{V}(p/N)\left[b_{p+q}^* a_{-p}^* a_q + a_q^* a_{-p} b_{p+q}\right] \\ & + \frac{\kappa}{2N} \sum_{p,q\in\Lambda_+^*, r\in\Lambda^*: r\neq -p,-q} \widehat{V}(r/N) a_{p+r}^* a_q^* a_p a_{q+r}\end{aligned}$$

Key fact: cubic and quartic terms cannot be neglected for $N\to\infty$

Step 2: include correlations between condensate and excitation pairs

Conjugation with U_N factors out products of the condensate wave function!

States with small energy in the Gross-Pitaevskii limit are characterized by a correlation structure on length scales of order N^{-1} which we model by the solution of the Neumann problem

$$\Big(-\Delta + \tfrac{\kappa}{2} N^2 V(Nx) \Big) f_{\ell,N}(x) = \lambda_{\ell,N} f_{N,\ell}(x)$$

on the ball $|x| \leq \ell < 1/2$, with

$$f_{N,\ell}(x) = 1 \quad \text{and} \quad \partial_{|x|} f_{N,\ell}(x) = 0 \qquad \text{for } |x| = \ell$$

One has

$$\Big| \underbrace{\kappa \int N^3 V(Nx) f_{\ell,N}(x) dx}_{\left(\kappa \widehat{V}(\cdot/N) \star \widehat{f}_{\ell,N}\right)_0} - 8\pi\mathfrak{a} \Big| \leq \frac{C\kappa}{N\ell}$$

Step 2: include correlations between condensate and excitation pairs

Inspired by [Brennecke-Schlein '17] we describe correlations in $\mathcal{F}_+^{\leq N}$ using

$$T(\eta) = \exp\left[\frac{1}{2}\sum_{p\in\Lambda_+^*}\eta_p\left(b_p^* b_{-p}^* - b_p b_{-p}\right)\right] : \mathcal{F}_+^{\leq N} \to \mathcal{F}_+^{\leq N}$$

with

$$\eta_p = -\tfrac{1}{N^2}\,\widehat{(1-f_{N,\ell})}(p/N)$$

and modified creation and annihilation operators

$$b_p^* = a_p^*\sqrt{\frac{N-\mathcal{N}}{N}}\,, \qquad b_p = \sqrt{\frac{N-\mathcal{N}}{N}}\,a_p \qquad : \mathcal{F}_+^{\leq N} \longrightarrow \mathcal{F}_+^{\leq N}$$

$$U_N^* b_p^* U_N = a_p^* \frac{a_0}{\sqrt{N}}\,, \qquad U_N^* b_p U_N = \frac{a_0^*}{\sqrt{N}} a_p \qquad : L^2(\Lambda^N) \longrightarrow L^2(\Lambda^N)$$

Remark: the operators b_p^* and b_p create and annihilate excitations, but do not change the total number of particles.

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$$|\eta_p| \leq C\frac{\kappa}{p^2}e^{-|p|/N}$$

$$\|\eta\|_2 \leq C\kappa, \;\; \|\eta\|_{H^1} \leq C\kappa\sqrt{N}$$

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Define the renormalized excitation Hamiltonian

$$\mathcal{G}_N = T^*(\eta) U_N H_N U_N^* T(\eta) \; : \; \mathcal{F}_+^{\leq N} \to \mathcal{F}_+^{\leq N}$$

Then

$$\mathcal{G}_N = 4\pi \mathfrak{a} N + \mathcal{H}_N + \theta_N \,, \qquad \pm \theta_N \leq \delta \mathcal{H}_N + C\kappa(\mathcal{N}_+ + 1)$$

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With this bound we prove that states s.t. $\langle \psi_N, H_N \psi_N \rangle \leq 4\pi \mathfrak{a} N + K$

can be written as $\quad \psi_N = U_N^* T(\eta) \xi_N \quad$ with $\quad \langle \xi_N, \mathcal{N}_+ \xi_N \rangle \leq C(K+1)$

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Refined bound: excitations associated to $\psi_N = \chi(H_N \leq 4\pi \mathfrak{a} N + K)\psi_N$ satisfy

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Moreover,

$$\mathcal{G}_N = C_{\mathcal{G}_N} + \mathcal{Q}_{\mathcal{G}_N} + \mathcal{C}_N + \mathcal{V}_N + \mathcal{E}_{\mathcal{G}_N} \,, \quad \pm \mathcal{E}_{\mathcal{G}_N} \leq C\, N^{-1/2} (\mathcal{H}_N + \mathcal{N}_+^2 + 1)(\mathcal{N}_+ + 1)$$

with $\mathcal{C}_N = \frac{\kappa}{\sqrt{N}} \sum_{p,q \in \Lambda_+^* : p+q \neq 0} \widehat{V}(p/N) \big(b_{p+q}^* a_{-p}^* a_q + \mathrm{h.c.} \big)$

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$$\psi_N = U_N^* T(\eta) \xi$$

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Not surprising! Quasi-free states can only approximate the ground state energy up to errors of order one [Erdoes-Schlein-Yau '08], [Napiorkovski-Reuvers-Solovej '15]. From RG perspective: quadratic and cubic terms are relevant in the ultraviolet.

Step 3: include correlations due to triplets

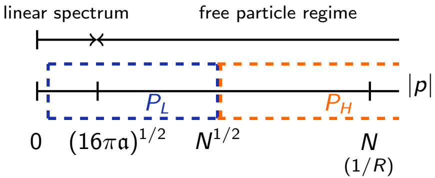
To extract the large term from $\mathcal{C}_N$ and $\mathcal{V}_N$ we consider the cubic operator

$$A = \frac{\kappa}{\sqrt{N}} \sum_{r \in P_H, v \in P_L} \eta_r \Big[b^*_{r+v} b^*_{-r} \left(\cosh(\eta)_v b_v + \sinh(\eta)_v b^*_{-v}\right) - \mathrm{h.c.} \Big]$$

with

$$P_L = \{p \in \Lambda^*_+ : |p| \leq N^{1/2}\}$$

$$P_H = \Lambda^*_+ / P_L$$



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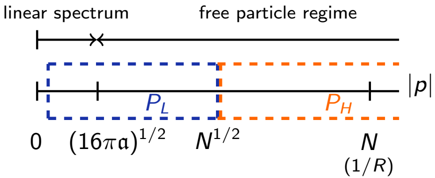

Hence, we define the renormalized excitation Hamiltonian

$$\mathcal{J}_N = e^{-A(\eta)} \mathcal{G}_N e^{A(\eta)} = e^{-A(\eta)} T^*(\eta) U_N \, H_N \, U_N^* T(\eta) e^{A(\eta)} \; : \; \mathcal{F}_+^{\leq N} \to \mathcal{F}_+^{\leq N}$$

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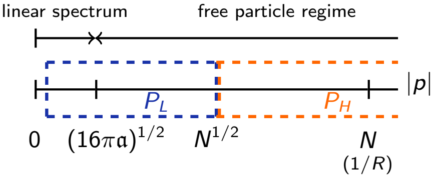

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Now,

$$\mathcal{J}_N = \underbrace{C_{\mathcal{J}_N} + \mathcal{Q}_{\mathcal{J}_N}}_{\text{determine the low energy spectrum}} + \mathcal{V}_N + \mathcal{E}_{\mathcal{J}_N}, \qquad \pm \mathcal{E}_{\mathcal{J}_N} \le C N^{-1/4} (\mathcal{H}_N + \mathcal{N}_+^2 + 1)(\mathcal{N}_+ + 1)$$

Step 3: include correlations due to triplets

Conjugation with the quadratic and cubic operators renormalizes the interaction, leading to the appearance of the scattering length:

$$\mathcal{J}_\mathcal{N} = \underbrace{C_{\mathcal{J}_N} + \mathcal{Q}_{\mathcal{J}_N}}_{\text{determine the low energy spectrum}} + \mathcal{V}_N + \mathcal{E}_{\mathcal{J}_N}, \qquad \pm \mathcal{E}_{\mathcal{J}_N} \leq C\, N^{-1/4}(\mathcal{N}_+ + 1)(\mathcal{H}_N + 1)$$

where

$$\mathcal{Q}_{\mathcal{J}_N} = \sum_{p \in \Lambda_+^*} \left[F_p b_p^* b_p + \frac{1}{2} G_p (\, b_p^* b_{-p}^* + b_p b_{-p}\,) \right]$$

with

$$F_p = p^2(\sinh^2\eta_p + \cosh^2\eta_p) + \kappa\left(\widehat{V}(\cdot/N) \star \widehat{f}_{N,\ell}\right)_p (\sinh\eta_p + \cosh\eta_p)^2$$

$$G_p = 2p^2 \sinh\eta_p \cosh\eta_p + \kappa\left(\widehat{V}(\cdot/N) \star \widehat{f}_{N,\ell}\right)_p (\sinh\eta_p + \cosh\eta_p)^2$$

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$$G_p = 2p^2 \sinh \eta_p \cosh \eta_p + \kappa \left(\widehat{V}(\cdot/N) \star \widehat{f}_{N,\ell} \right)_p (\sinh \eta_p + \cosh \eta_p)^2 \simeq \frac{1}{p^2}$$

The operator $\mathcal{Q}_{\mathcal{J}_N}$ may be diagonalized using

$$T(\tau) = \exp \left[\frac{1}{2} \sum_{p \in \Lambda_+^*} \tau_p (b_p^* b_{-p}^* - b_p b_{-p}) \right], \quad \tanh(2\tau_p) = -\frac{G_p}{F_p} \qquad |\tau_p| \simeq |p|^{-4}$$

Step4: diagonalization

Let $\mathcal{M}_N = T^*(\tau)\mathcal{J}_N\, T(\tau) : \mathcal{F}_+^{\leq N} \to \mathcal{F}_+^{\leq N}$, then

$$\mathcal{M}_N = E_N + \sum_{p\in\Lambda_+^*} \sqrt{|p|^4 + 16\pi\mathfrak{a}|p|^2}\, a_p^* a_p + \mathcal{E}_{\mathcal{M}_N}$$

with

$$E_N = 4\pi(N-1)\mathfrak{a}_N - \frac{1}{2}\sum_{p\in\Lambda_+^*}\left[p^2 + 8\pi\mathfrak{a} - \sqrt{|p|^4 + +16\pi\mathfrak{a}|p|^2} + \frac{(8\pi\mathfrak{a})^2}{2p^2}\right]$$

and

$$\mathcal{E}_{\mathcal{M}_N} \leq CN^{-1/4}(\mathcal{H}_N + \mathcal{N}_+^2 + 1)(\mathcal{N}_+ + 1).$$

Finally, we use of the min-max principle to compare the eigenvalues λ_m of $\mathcal{M}_N - E_N$ (*i.e.* the eigenvalues of $H_N - E_N$) with the eigenvalues $\tilde{\lambda}_m$ of

$$\mathcal{D}_N = \sum_{p\in\Lambda_+^*} \sqrt{|p|^4 + +16\pi\mathfrak{a}|p|^2}\, a_p^* a_p\,,$$

showing that below an energy ζ

$$|\lambda_m - \tilde{\lambda}_m| \leq C\, N^{-1}(1+\zeta^3)$$

Perspectives

- Condensation and Bogoliubov theory in the Gross-Pitaevskii regime without the smallness condition on the interaction
- Extend the results to non-translation-invariant bosonic systems trapped by confining external fields
- Next term in the ground state energy expansion ?
- ...
- Validity of Bogoliubov predictions for dilute Bose gases in the thermodynamic limit
- Connection with Renormalization Group methods

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Juan J. L. Velázquez
(University of Bonn)

Speakers

Carlo Barenghi (Newcastle University)

Iacopo Carusotto (INO-CNR BEC Center & University of Trento)

Marco Falconi (University of Tübingen)

Evelyne Miot-Desecures (CNRS & Université Grenoble-Alpes)

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