Telegraph equation from the six-vertex model

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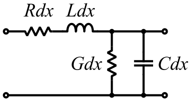
Telegraph equation

Resistance R Inductance L Capacitance C Conductance G



Wiki/Pixabay

Voltage $\mathbf{V}$ Current $\mathbf{I}$

$$\frac{\partial \mathbf{V}}{\partial x}(x,t) = -L \cdot \frac{\partial \mathbf{I}}{\partial t}(x,t) - R \cdot \mathbf{I}(x,t)$$

$$\frac{\partial \mathbf{I}}{\partial x}(x,t) = -C \cdot \frac{\partial \mathbf{V}}{\partial t}(x,t) - G \cdot \mathbf{V}(x,t)$$

or

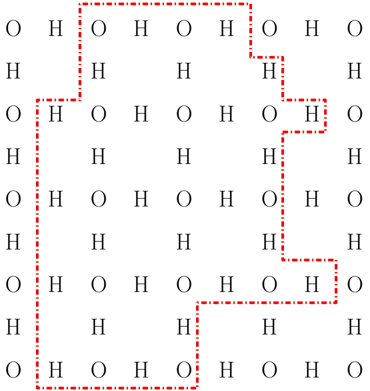
$$\boxed{\underbrace{\mathbf{V}_{xx} - LC \cdot \mathbf{V}_{tt}}_{\text{Wave equation}} - \underbrace{(RC + GL) \cdot \mathbf{V}_t - GR \cdot \mathbf{V}}_{\text{Effect of losses}} = 0}$$

Six–vertex model

O	H	O	H	O	H	O	H	O
H		H		H		H		H
O	H	O	H	O	H	O	H	O
H		H		H		H		H
O	H	O	H	O	H	O	H	O
H		H		H		H		H
O	H	O	H	O	H	O	H	O
H		H		H		H		H
O	H	O	H	O	H	O	H	O

Square grid with O in the vertices and H on the edges.

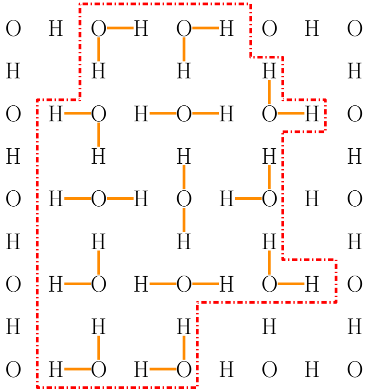
Six–vertex model



Square grid with O in the vertices and H on the edges.

Finite/infinite domain.

Six–vertex model



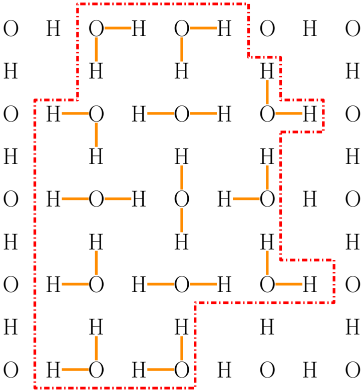
Square grid with O in the vertices and H on the edges.

Finite/infinite domain.

Configurations: possible matchings of *all* atoms inside domain into H_2O molecules.

This is **square ice model**.
Real-world ice has somewhat similar (although 3d) structure.

Six–vertex model



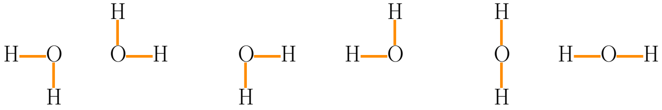
Square grid with O in the vertices and H on the edges.

Finite/infinite domain.

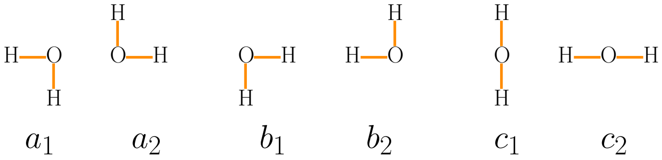
Configurations: possible matchings of *all* atoms inside domain into H_2O molecules.

This is square ice model.
Real-world ice has somewhat similar (although 3d) structure.

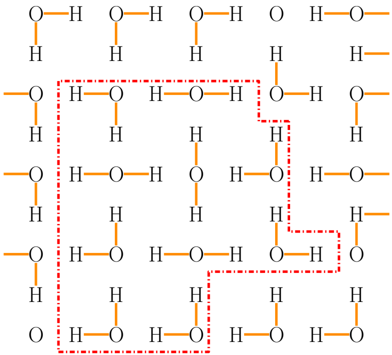
Also known as six vertex model.



Six–vertex model: Gibbs measures



Statistical mechanics starting from (Lieb–67):



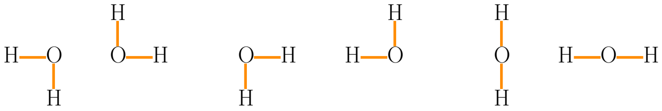
Assign Gibbs weights

$$\frac{a_1^{\#(a_1)} a_2^{\#(a_2)} b_1^{\#(b_1)} b_2^{\#(b_2)} c_1^{\#(c_1)} c_2^{\#(c_2)}}{Z(a_1, a_2, b_1, b_2, c_1, c_2)}$$

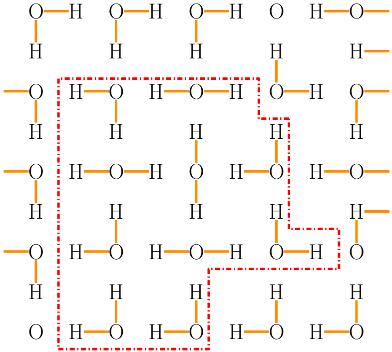
[Depends only on $\frac{b_1 b_2}{a_1 a_2}$ and $\frac{c_1 c_2}{a_1 a_2}$.]

Asymptotic properties of Gibbs measures?

Six–vertex model: Gibbs measures



a_1 a_2 b_1 b_2 c_1 c_2



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[Depends only on $\frac{b_1 b_2}{a_1 a_2}$ and $\frac{c_1 c_2}{a_1 a_2}$.]

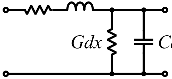
Asymptotic **properties of Gibbs measures?**

Understanding is still **very limited.**

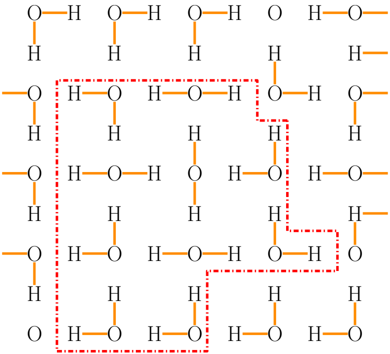
Question for today

Telegraph equation

$$\mathbf{V}_{xx} - \mathbf{V}_{tt} - \alpha \mathbf{V}_t - \beta \mathbf{V}_x - \gamma \mathbf{V} = 0$$



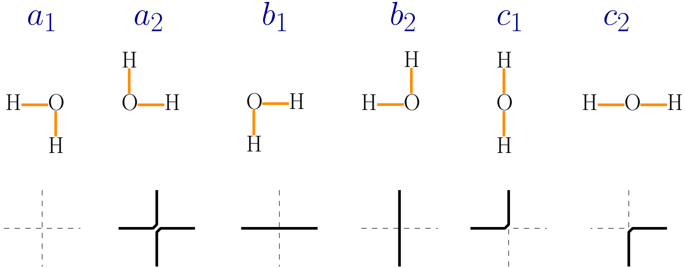
Six–vertex model



$$\frac{a_1^{\#(a_1)} a_2^{\#(a_2)} b_1^{\#(b_1)} b_2^{\#(b_2)} c_1^{\#(c_1)} c_2^{\#(c_2)}}{Z(a_1, a_2, b_1, b_2, c_1, c_2)}$$

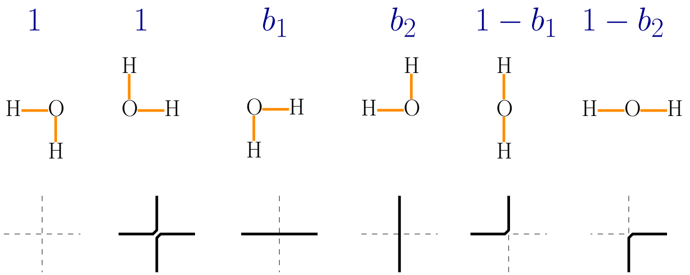
What do they have in common?

Stochastic six–vertex model



An equivalent representation
Collection of **paths** on the plane

Stochastic six–vertex model

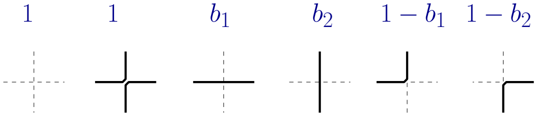


Assumption:
(Gwa–Spohn–92)

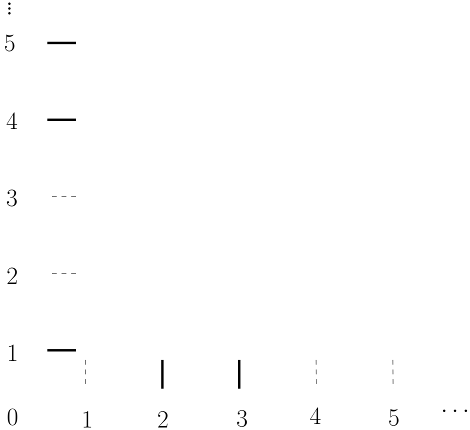
$$a_1 = a_2 = 1, \quad b_1 + c_1 = 1, \quad b_2 + c_2 = 1$$

$$\left(\text{Implies} \quad \Delta = \frac{a_1 a_2 + b_1 b_2 - c_1 c_2}{2\sqrt{a_1 a_2 b_1 b_2}} \geq 1\right)$$

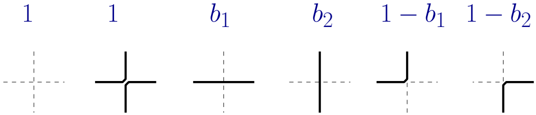
Stochastic six–vertex model



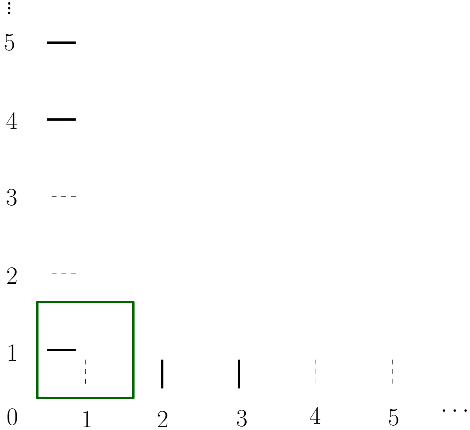
Take arbitrary **boundary conditions** in the quadrant



Stochastic six–vertex model

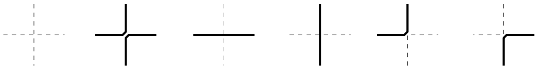


Proceed with **sequential stochastic sampling**



Stochastic six–vertex model

$1 \quad 1 \quad b_1 \quad b_2 \quad 1-b_1 \quad 1-b_2$



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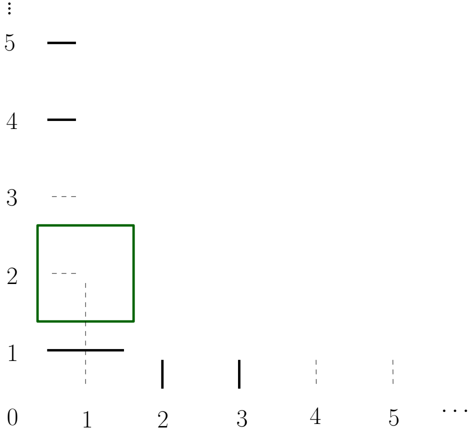

no choice

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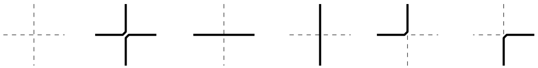

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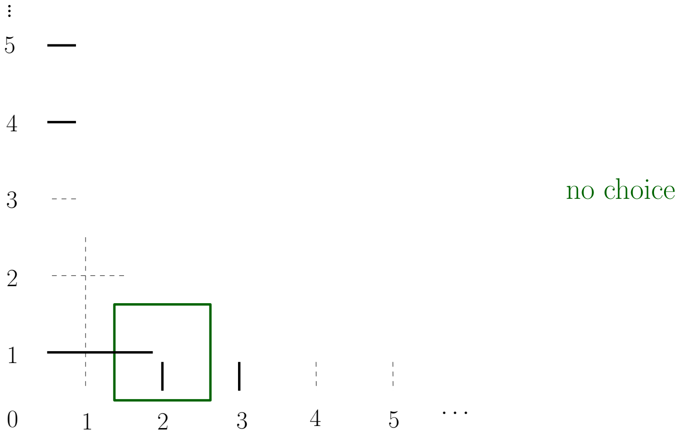

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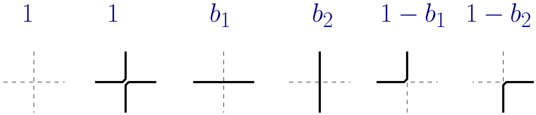

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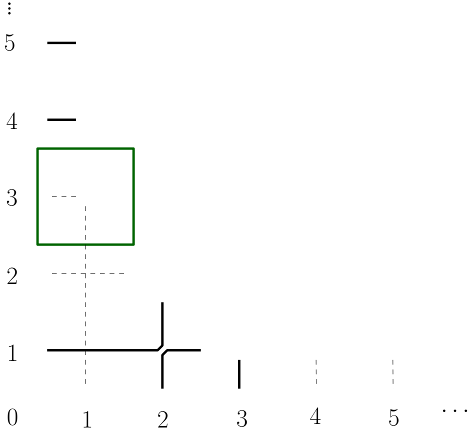

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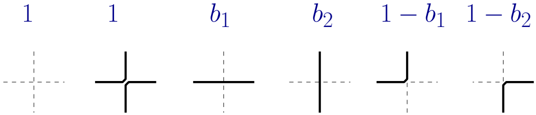

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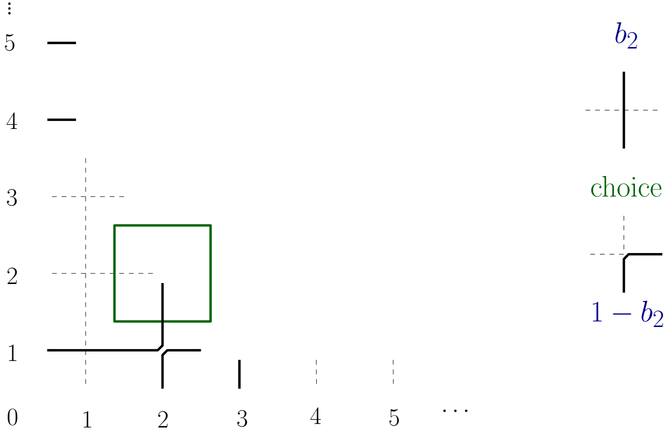

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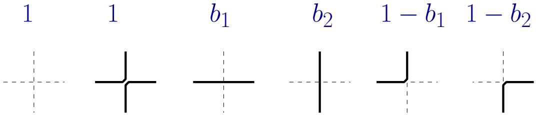

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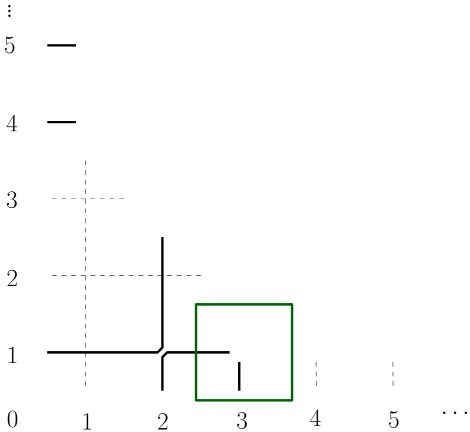

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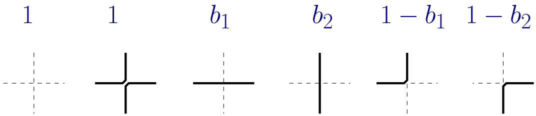

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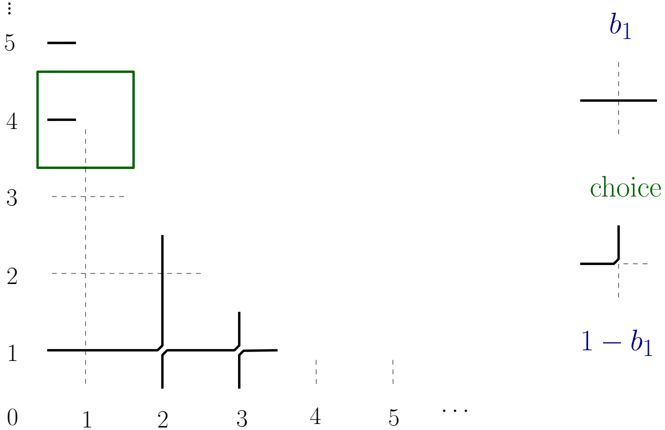

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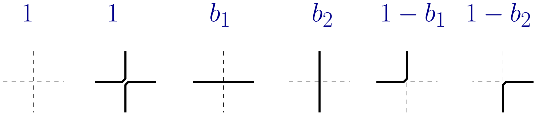

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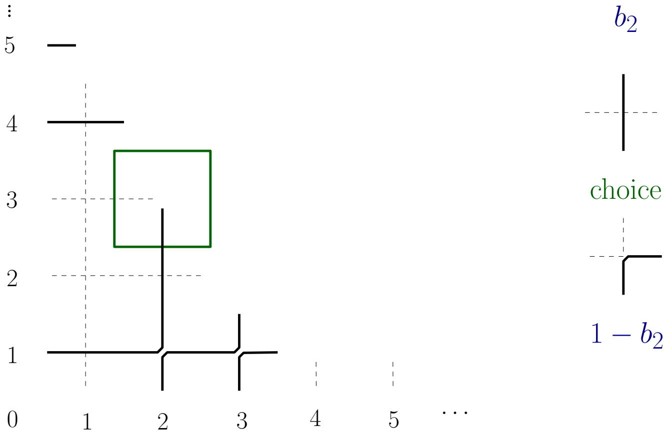

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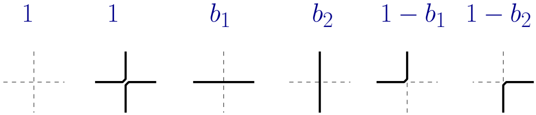

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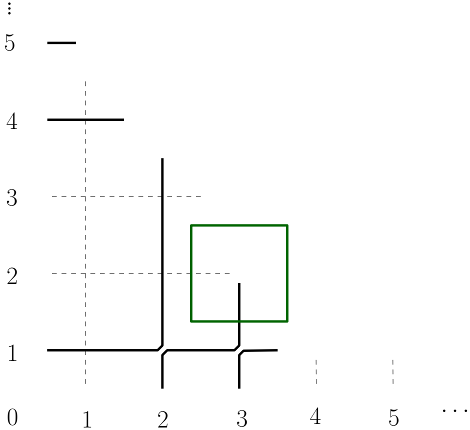
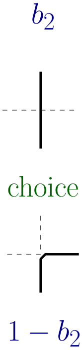
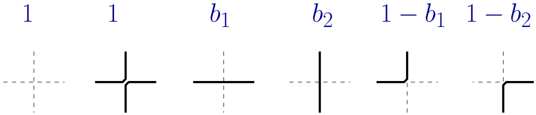




Stochastic six–vertex model



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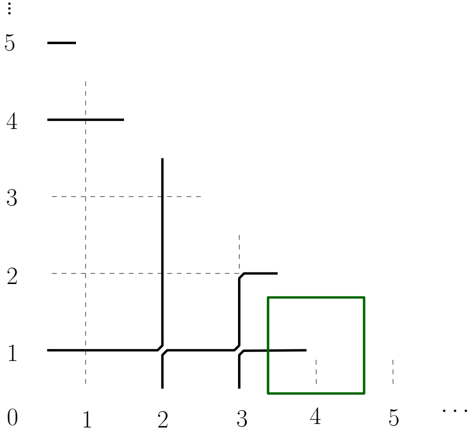

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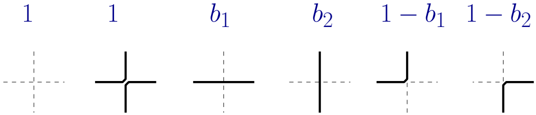
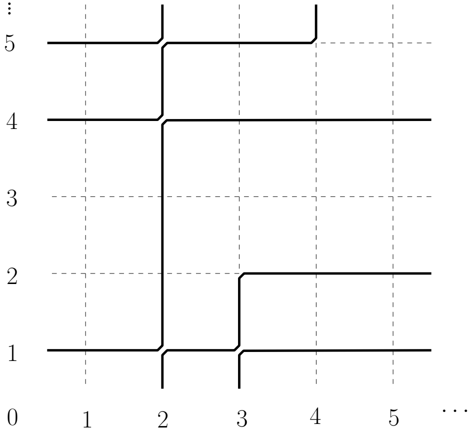


Until the quadrant is **filled**



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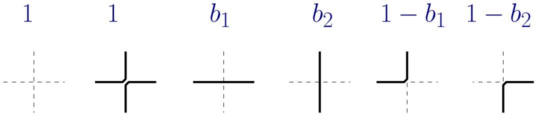
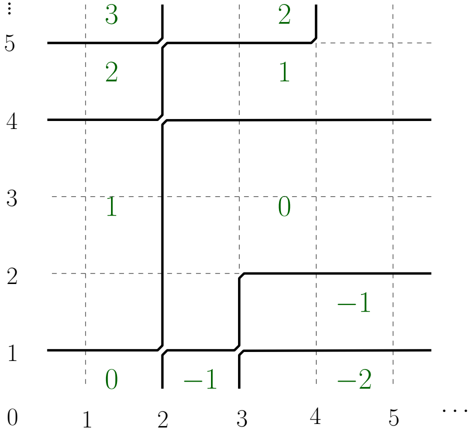
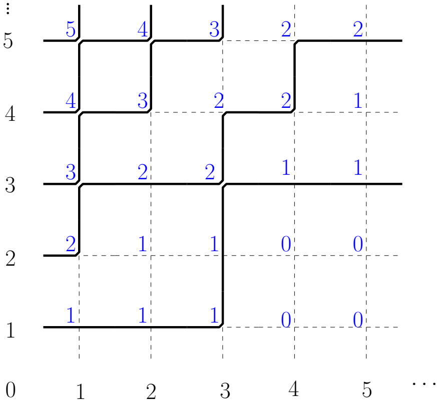


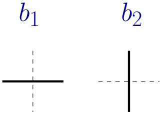
The resulting paths are level lines of the **height function**



- Height is 0 at the origin,
- increases up,
- decreases to the right.

Domain–wall and fixed weights





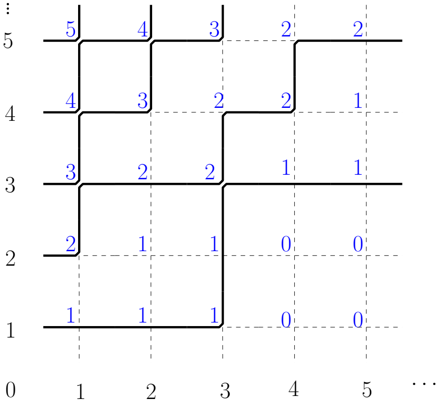
$$\boxed{\mathfrak{s} = \frac{1-b_1}{1-b_2}}$$

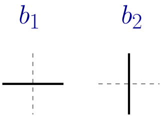
$$\frac{1}{L} H(Lx, Ly) \to \mathfrak{h}(x,y) = \begin{cases} 0, & \frac{x}{y} > \mathfrak{s}^{-1}, \\ \dfrac{(\sqrt{\mathfrak{s}x} - \sqrt{y})^2}{1-\mathfrak{s}}, & \mathfrak{s} \le \frac{x}{y} \le \mathfrak{s}^{-1} \\ y - x, & \frac{x}{y} < \mathfrak{s}. \end{cases}$$

Theorem. (Borodin–Corwin–Gorin-14) For domain–wall boundary conditions and fixed $0 < b_2 < b_1 < 1$, $\frac{1}{L} H(Lx, Ly) \to \mathfrak{h}(x,y)$ with fluctuations on $L^{1/3}$ scale given by the Tracy–Widom distribution.

TW = universal law for the largest eigenvalue of Hermitian matrices
and for particle system in Kardar–Parisi–Zhang class

Domain–wall and fixed weights





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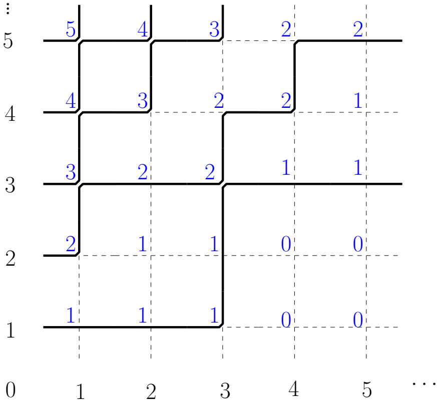
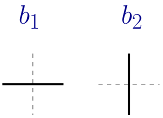
$$\rho_x \cdot \mathfrak{s} + \rho_y \cdot (\mathfrak{s} + (\mathfrak{s}-1)\rho)^2 = 0, \qquad \rho = \mathfrak{h}_x.$$

1st–order non-linear PDE (Gwa–Spohn–92) (Reshetikhin–Sridhar–16)

Domain–wall and fixed weights





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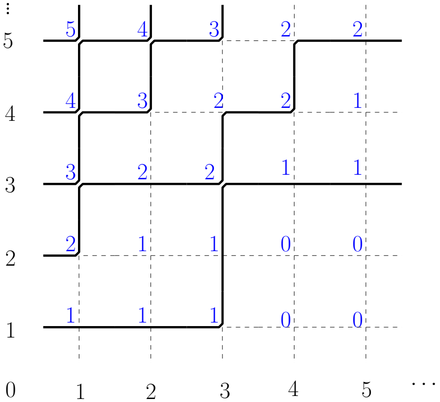
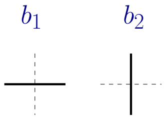
- Instead of b_1, b_2, only $\mathfrak{s}$. Where is the second parameter?
- Where is the linear telegraph equation?

Domain–wall and fixed weights





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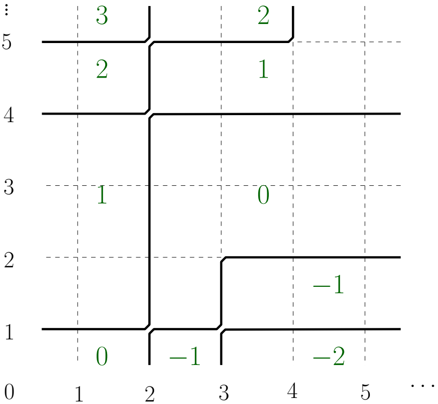
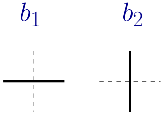
- Instead of b_1, b_2, only $\mathfrak{s}$. Where is the second parameter?
- Where is the linear telegraph equation?

(Borodin–Gorin–18): One needs to rescale weights b_1, b_2.

Rescaled weights: Law of Large Numbers





Low density of corners

$$b_1 = \exp\left(-\frac{\beta_1}{L}\right)$$

$$b_2 = \exp\left(-\frac{\beta_2}{L}\right)$$

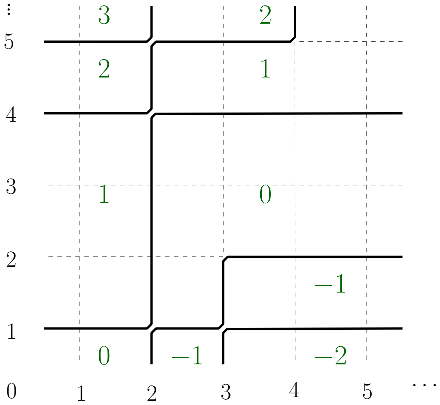
$$\mathfrak{q} = \left(\frac{b_2}{b_1}\right)^L = e^{\beta_1 - \beta_2}$$

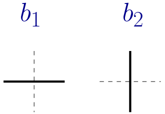
Theorem. (Borodin–Gorin–18) For **arbitrary** boundary conditions and **rescaled weights**, $\frac{1}{L} H(Lx, Ly) \to \mathfrak{h}(x, y)$ with

$$\frac{\partial^2}{\partial x \partial y}\left(\mathfrak{q}^{\mathfrak{h}(x,y)}\right) + \beta_2 \frac{\partial}{\partial x}\left(\mathfrak{q}^{\mathfrak{h}(x,y)}\right) + \beta_1 \frac{\partial}{\partial y}\left(\mathfrak{q}^{\mathfrak{h}(x,y)}\right) = 0,$$

$$\mathfrak{q}^{\mathfrak{h}(x,0)} = \chi(x), \quad \mathfrak{q}^{\mathfrak{h}(0,y)} = \psi(y).$$

Rescaled weights: Law of Large Numbers





Low density of corners

$$b_1 = \exp\left(-\frac{\beta_1}{L}\right)$$

$$b_2 = \exp\left(-\frac{\beta_2}{L}\right)$$

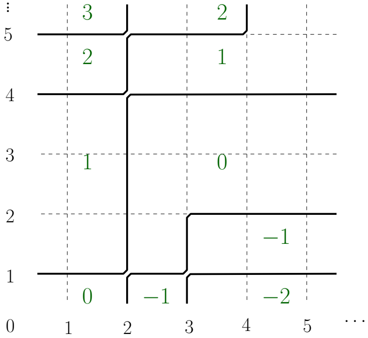
$$\mathfrak{q} = \left(\frac{b_2}{b_1}\right)^L = e^{\beta_1 - \beta_2}$$

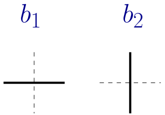
$$\left(\mathfrak{q}^{\mathfrak{h}(x,y)}\right)_{xy} + \beta_2 \left(\mathfrak{q}^{\mathfrak{h}(x,y)}\right)_x + \beta_1 \left(\mathfrak{q}^{\mathfrak{h}(x,y)}\right)_y = 0$$

$$\mathfrak{q}^{\mathfrak{h}(x,0)} = \chi(x), \quad \mathfrak{q}^{\mathfrak{h}(0,y)} = \psi(y).$$

- In $t = x + y$, $z = x - y$, a version of the **Telegraph equation**.
- **Characteristic** Cauchy problem has a unique solution.

Rescaled weights: Law of Large Numbers





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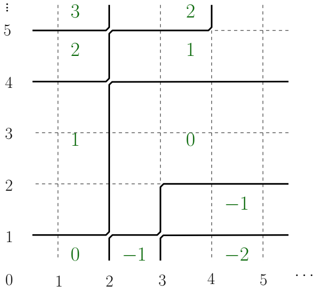
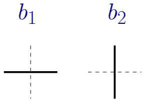
2nd order **hyperbolic** limit shape equation is **strange**:

- Diffusions: **parabolic** equations (e.g. Brownian motion)
- Interacting particle systems: **1st order** equations (e.g. TASEP)
- $2d$ statistical mechanics: 2nd order **elliptic** Euler–Lagrange equations through variational principles (e.g. random tilings)

Rescaled weights: Fluctuations





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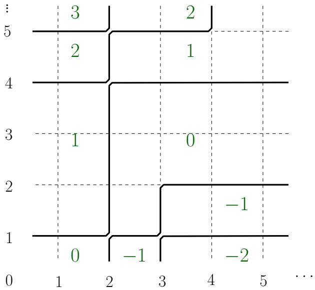
$$q = \frac{b_2}{b_1} = \mathfrak{q}^{1/L}$$

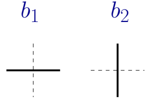
$$\frac{1}{L} H(Lx, Ly) \to \mathfrak{h}(x, y).$$

What about **fluctuations** $H(Lx, Ly) - \mathbb{E}H(Lx, Ly)$?

Reminder. For **fixed** b_1, b_2, they were Tracy–Widom on $L^{1/3}$ scale.

Rescaled weights: Fluctuations





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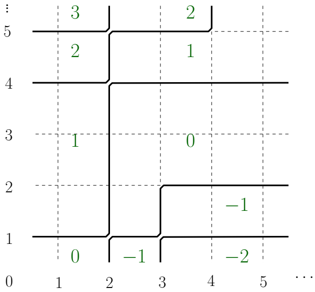
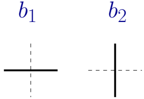
Claim. $H(Lx, Ly) - \mathbb{E}H(Lx, Ly) \approx L^{1/2} \times$ Gaussian.

Rescaled weights: Fluctuations





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Claim. $H(Lx, Ly) - \mathbb{E}H(Lx, Ly) \approx L^{1/2} \times$ Gaussian.

Theorem. (Borodin–Gorin–18, Shen–Tsai–18)

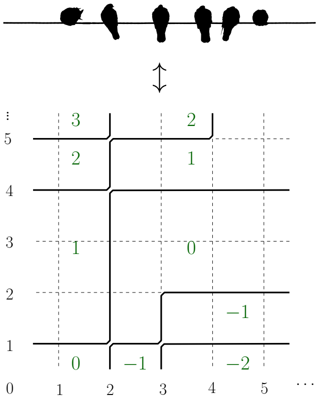
$\lim\limits_{L\to\infty} \sqrt{L}\left(q^{H(Lx,Ly)} - \mathbb{E}q^{H(Lx,Ly)}\right)$ solves **Stochastic Telegraph**

$$\phi_{xy} + \beta_2\phi_x + \beta_1\phi_y = \dot{W}\sqrt{V(x,y)}$$

$$V(x,y) = (\beta_1+\beta_2)\mathfrak{q}_x^{\mathfrak{h}}\mathfrak{q}_y^{\mathfrak{h}} + (\beta_2-\beta_1)\beta_2\mathfrak{q}^{\mathfrak{h}}\mathfrak{q}_x^{\mathfrak{h}} - (\beta_2-\beta_1)\beta_1\mathfrak{q}^{\mathfrak{h}}\mathfrak{q}_y^{\mathfrak{h}}$$

R.H.S.=2d white noise $\times$ non-linear functional of the limit shape

Six–vertex and Telegraph



Stochastic six–vertex model in the quadrant in low corner density asymptotic regime.

- Deterministic limit (LLN) for $q^{H(x,y)}$ is given by the homogeneous Telegraph equation.
- Gaussian fluctuations (CLT) are given by Stochastic Telegraph equation.

Why?

Feynman–Kac for Heat equation

For a second, switch to (parabolic) **Heat equation**.

$$H_t = \frac{1}{2} H_{xx},\ t \geq 0, \qquad H(0,x) = f(x).$$

The Feynman–Kac formula expresses the solution:

$$H(t,x) = \mathbb{E} f(B_t),$$

where B_t is the **Brownian motion** started at $B_0 = x$.

Similar representation is possible for the Telegraph equation!

Feynman–Kac for Telegraph

Persistent random walk.

Turns down/to the left at Poisson random times.

Feynman–Kac for Telegraph

Persistent random walk.

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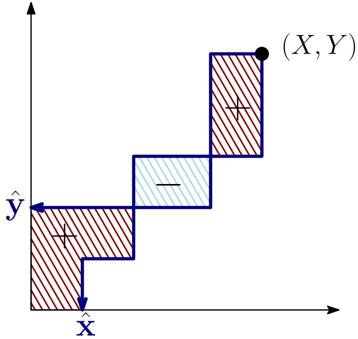


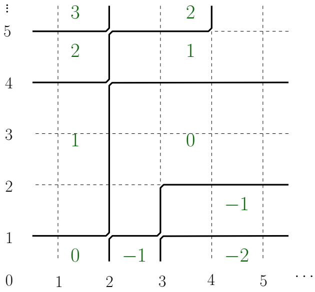
Theorem. (Borodin–Gorin–18; following Goldstein–51, Kac–74)

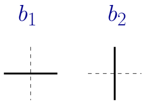
$$\phi_{xy} + \beta_2 \phi_x + \beta_1 \phi_y = u, \quad x, y > 0; \quad \phi(x, 0) = \chi(x), \ \phi(0, y) = \psi(y).$$

Then random characteristics solve the inhomogeneous Telegraph:

$$\phi(X, Y) = \mathbb{E}\chi(\hat{\mathbf{x}}) + \mathbb{E}\psi(\hat{\mathbf{y}}) + \mathbb{E}\left[\int_0^X \int_0^Y \mathcal{I}^{\pm}_{\text{between}}(x, y) u(x, y) dx dy\right].$$

Six–vertex and persistent random walks

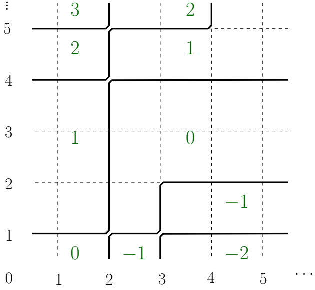


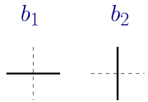


$$b_1 = \exp\left(-\frac{\beta_1}{L}\right) \quad b_2 = \exp\left(-\frac{\beta_2}{L}\right)$$

If paths are **rare** (low density limit), then each of them becomes a persistent random walk. They are essentially independent.

Six–vertex and persistent random walks





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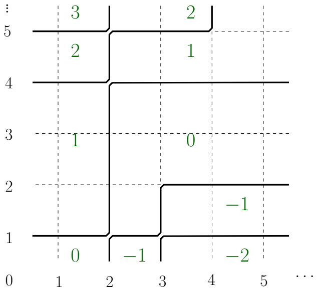
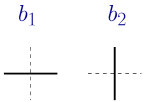
Conclusion. LLN/CLT for the height at **low density** is the same as LLN/CLT for a family of independent persistent random walks. Hence, connection to the Telegraph equation.

Six–vertex and persistent random walks





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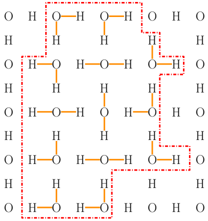
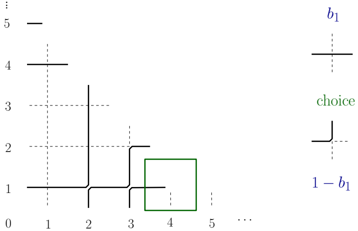
Conclusion. LLN/CLT for the height at **low density** is the same as LLN/CLT for a family of independent persistent random walks. Hence, connection to the Telegraph equation.

How to explain the same connection at **high densities** and the appearance of $q^{H(x,y)}$?

Summary



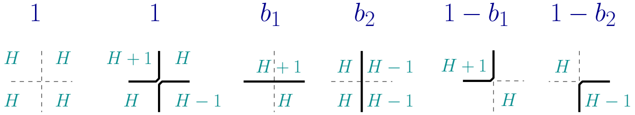
$$\phi_{xy} + \beta_2 \phi_x + \beta_1 \phi_y = \dot{W}\sqrt{V}$$



- Stochastic **six–vertex model** in the rare corners regime unexpectedly connects to **hyperbolic PDEs**.
- $\lim_{L\to\infty} q^{H(Lx,Ly)}$ solves Telegraph equation.
- $\mathfrak{q} \to 0$: fixed weights 1st order nonlinear PDE for LLN.
- $\lim_{L\to\infty} \sqrt{L}(q^H - \mathbb{E}q^H)$ — Stochastic Telegraph.
- Links to persistent random walks — Feynman–Kac formula for Telegraph.

Main tool: four point relation

Proofs rely on **exact discrete analogue** of stochastic Telegraph.



Theorem. (Borodin–Gorin–18; with help of Wheeler) For the stochastic six–vertex model in the quadrant with **arbitrary** boundary conditions, and each $x, y = 1, 2, \dots$, set for $q = \frac{b_2}{b_1}$:

$$\xi(x,y) = q^{H(x,y)} - b_1 q^{H(x-1,y)} - b_2 q^{H(x,y-1)} + (b_1 + b_2 - 1) q^{H(x-1,y-1)}.$$

Then ξ is a **martingale** with explicit variance:

1. $\mathbb{E}\left[\xi(x,y) \mid H(u,v), u < x \text{ or } v < y\right] = 0.$
2. $\mathbb{E}\left[\xi^2(x,y) \mid H(u,v), u < x \text{ or } v < y\right] =$

$$\big(b_1(1-b_1) + b_1(1-b_2)\big)\Delta_x \Delta_y + b_1(1-b_2)(1-q) q^{H(x,y)} \Delta_x - b_1(1-b_1)(1-q) q^{H(x,y)} \Delta_y,$$

with $\quad \Delta_x = q^{H(x,y-1)} - q^{H(x-1,y-1)}, \qquad \Delta_y = q^{H(x-1,y)} - q^{H(x-1,y-1)}$