Schrödinger Operators With Thin Spectra

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Introduction

In this talk we discuss the spectrum $\sigma(H_V)$ of a Schrödinger operator $H_V = -\Delta + V$ in $L^2(\mathbb{R}^d)$.

If the potential V vanishes identically, then the spectrum is a half-line, $\sigma(H_0) = [0, \infty)$.

If the potential V is periodic, then the spectrum $\sigma(H_V)$ is a union of non-degenerate intervals.

If either of these cases is perturbed by a perturbation vanishing at infinity, the spectrum may additionally have isolated points.

Introduction

Notice that in the scenarios above, the spectrum consists of intervals and isolated points.

In one of the major developments in the spectral theory of Schrödinger operators in the 1980's it was realized that (even for quite reasonable potentials), the spectrum can be such that it neither has any isolated points nor contains any intervals — i.e., it is a (generalized) Cantor set.

Let us present and elucidate some recent results that go further in the direction of "thin spectra."

All of these results concern the one-dimensional case, i.e. operators of the form $H_V = -\frac{d^2}{dx^2} + V$ in $L^2(\mathbb{R})$.

Zero-Measure Spectrum via a Fibonacci Structure

The (discrete) Fibonacci Hamiltonian is the bounded self-adjoint operator

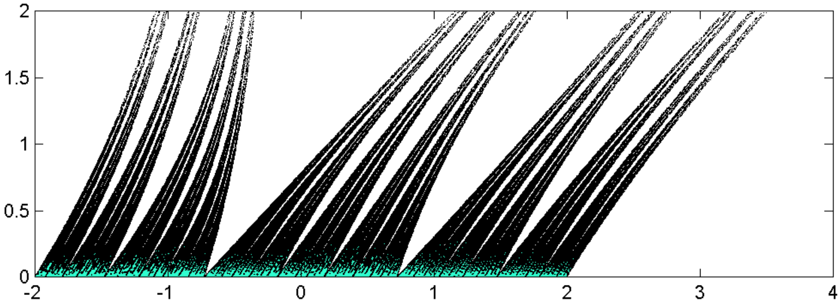
$$[H^{(\mathrm{Fib})}_{\lambda,\omega}\psi](n) = \psi(n+1) + \psi(n-1) + \lambda\chi_{[1-\alpha,1)}(n\alpha + \omega \mod 1)\psi(n)$$

in $\ell^2(\mathbb{Z})$, with the coupling constant $\lambda > 0$ and the phase $\omega \in \mathbb{T}$. The frequency is given by $\alpha = \frac{\sqrt{5}-1}{2}$. This operator has been studied in a large number of papers since the early 1980's.

Theorem (Sütő 1989)

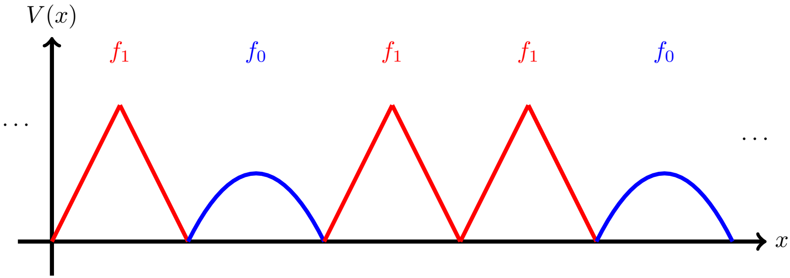
For every $\lambda > 0$, the ω-independent spectrum of $H^{(\mathrm{Fib})}_{\lambda,\omega}$ is a Cantor set of zero Lebesgue measure.

The Spectrum in the Discrete Case



The Continuum Fibonacci Hamiltonian

The continuum counterpart was studied by Damanik, Fillman and Gorodetski in a 2014 AHP paper. It replaces the two-valued sequence by an analogous sequence of "bumps" of two types, f_0 and f_1:



The Continuum Fibonacci Hamiltonian

We need to assume a non-degeneracy condition, such as the aperiodicity of the resulting continuum potential V.

Theorem (D.-Fillman-Gorodetski 2014)

Under the non-degeneracy assumption, the spectrum of H_V is a generalized Cantor set of zero Lebesgue measure.

Remarks. (a) By a generalized Cantor set we mean a closed nowhere dense set without isolated points.

(b) The non-degeneracy assumption clearly cannot be dropped.

(c) The proof gives information about the (local and global) Hausdorff dimension of the spectrum.

The Trace Map Formalism

The key to this result (and in particular to some of its quantitative companion results not discussed explicitly here) is a sophisticated application of hyperbolic dynamics to the study of the Fibonacci *trace map*, which is given by

$$T : \mathbb{R}^3 \to \mathbb{R}^3, \ T(x, y, z) = (2xy - z, x, y)$$

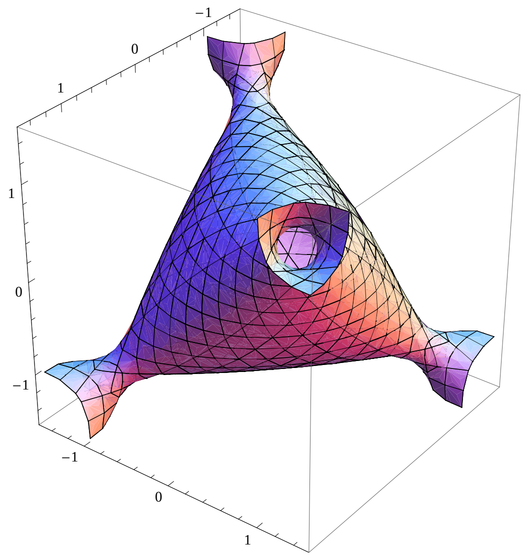
The function

$$I(x, y, z) = x^2 + y^2 + z^2 - 2xyz - 1$$

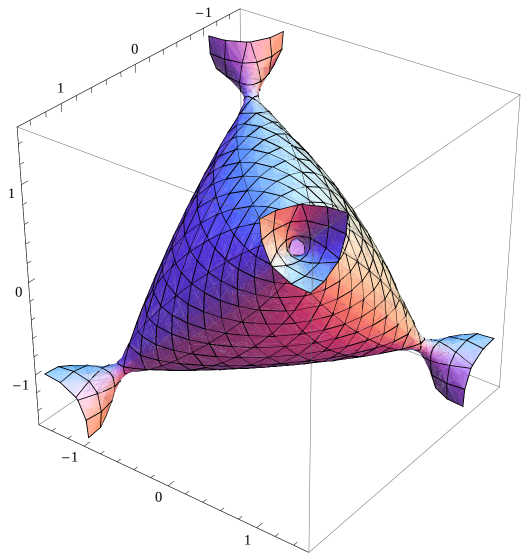
is invariant under the action of T and hence T preserves the surfaces

$$S_I = \{(x, y, z) \in \mathbb{R}^3 : I(x, y, z) = I\}$$

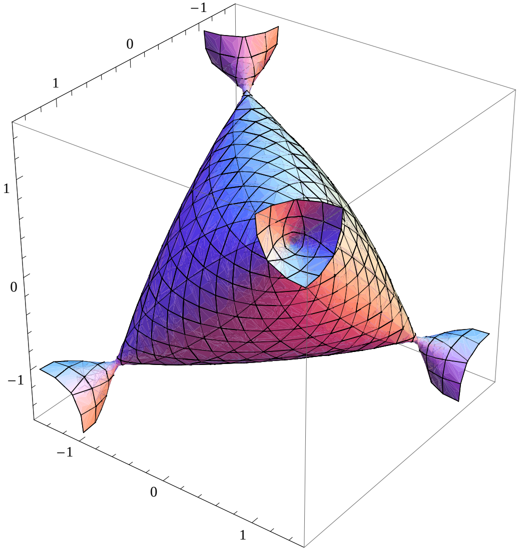
The Surface $S_{0.5}$



The Surface $S_{0.2}$



The Surface $S_{0.1}$



The Trace Map as a Surface Diffeomorphism

It is therefore natural to consider the restriction T_I of the trace map T to the invariant surface S_I. That is, $T_I : S_I \to S_I$, $T_I = T|_{S_I}$.

We denote by Λ_I the set of points in S_I whose full orbits under T_I are bounded.

Denote by ℓ_λ the line

$$\ell_\lambda = \left\{ \left(\frac{E-\lambda}{2}, \frac{E}{2}, 1 \right) : E \in \mathbb{R} \right\}$$

It is easy to check that $\ell_\lambda \subset S_{\frac{\lambda^2}{4}}$.

Spectrum and Bounded Trace Map Orbits

The key to the fundamental connection between the spectral properties of the Fibonacci Hamiltonian and the dynamics of the trace map is the following result:

Proposition (Sütő 1987)

An energy $E \in \mathbb{R}$ belongs to the spectrum of the discrete Fibonacci Hamiltonian $H_{\lambda,\omega}^{(\mathrm{Fib})}$ if and only if the positive semiorbit of the point $(\frac{E-\lambda}{2}, \frac{E}{2}, 1)$ under iterates of the trace map T is bounded.

Λ_λ is a Locally Maximal Hyperbolic Set

Let us recall that an invariant closed set Λ of a diffeomorphism $f : M \to M$ is *hyperbolic* if there exists a splitting of the tangent space $T_x M = E^s_x \oplus E^u_x$ at every point $x \in \Lambda$ such that this splitting is invariant under Df, the differential Df exponentially contracts vectors from the stable subspaces $\{E^s_x\}$, and the differential of the inverse, Df^{-1}, exponentially contracts vectors from the unstable subspaces $\{E^u_x\}$.

A hyperbolic set Λ of a diffeomorphism $f : M \to M$ is *locally maximal* if there exists a neighborhood U of Λ such that

$$\Lambda = \bigcap_{n \in \mathbb{Z}} f^n(U)$$

It is known (Casdagli 1986, Damanik-Gorodetski 2009, Cantat 2009) that for $I > 0$, the set Λ_I is a locally maximal hyperbolic set of $T_I : S_I \to S_I$.

The Continuum Case

The existence of the trace map (and as a consequence, the existence of the invariant, the restrictions to invariant surfaces, and the Markov partitions) is solely a consequence of the self-similarity of the discrete Fibonacci sequence.

Since the continuum potential inherits this self-similarity, all the resulting objects continue to exist and are the same as before.

The primary difference between the discrete and the continuum case is seen in the curve of initial conditions (which is given by the line ℓ_λ in the discrete case). Let us recall how the line ℓ_λ arises and what it is replaced with in the continuum case.

The Continuum Case

The continuum model depends on choices of lengths $\ell_0, \ell_1 > 0$ and real-valued functions $f_0 \in L^2(0, \ell_0)$ and $f_1 \in L^2(0, \ell_1)$, the *local potentials*.

Then the potential of the Schrödinger operator H in question is obtained by piecing together translates of the local potentials according to the Fibonacci sequence

$$v_F(n) = \chi_{[1-\alpha,1)}(n\alpha \bmod 1); \quad n \in \mathbb{Z},\ \alpha = \frac{\sqrt{5}-1}{2}$$

Recall that we impose a non-degeneracy assumption.

The Curve of Initial Conditions

Consider the solutions of the differential equation

$$-u''(x) + f_0(x)u(x) = Eu(x)$$

for real energy E.

Denote the solution obeying $u(0) = 0$, $u'(0) = 1$ (resp., $u(0) = 1$, $u'(0) = 0$) by $u_{0,D}(\cdot, E)$ (resp., $u_{0,N}(\cdot, E)$). Similarly, by replacing f_0 with f_1, we define $u_{1,D}(\cdot, E)$ and $u_{1,N}(\cdot, E)$.

Then, we set

$$M_0(E) = \begin{pmatrix} u_{0,N}(\ell_0, E) & u_{0,D}(\ell_0, E) \\ u'_{0,N}(\ell_0, E) & u'_{0,D}(\ell_0, E) \end{pmatrix}$$

$$M_1(E) = \begin{pmatrix} u_{1,N}(\ell_1, E) & u_{1,D}(\ell_1, E) \\ u'_{1,N}(\ell_1, E) & u'_{1,D}(\ell_1, E) \end{pmatrix}$$

The Curve of Initial Conditions

Moreover, let

$$x_0(E) = \frac{1}{2}\mathrm{tr}\,(M_0(E))$$
$$x_1(E) = \frac{1}{2}\mathrm{tr}\,(M_1(E))$$
$$x_2(E) = \frac{1}{2}\mathrm{tr}\,(M_0(E)M_1(E))$$

The map $E \mapsto (x_2(E), x_1(E), x_0(E))$ will be called the *curve of initial conditions*, and this is the continuum replacement of the line of initial conditions that played a key role in the discrete case.

Spectrum and Dynamical Spectrum

The points $T^n(x_2(E), x_1(E), x_0(E))$ lie on the surface $S_{I(E)}$, where (with some abuse of notation) we set

$$I(E) = I(x_2(E), x_1(E), x_0(E))$$

The *dynamical spectrum* B is defined by

$$B = \{E \in \mathbb{R} : \{T^n(x_2(E), x_1(E), x_0(E))\}_{n \in \mathbb{Z}_+} \text{ is bounded}\}$$

and it was shown to coincide with the spectrum of the continuum Fibonacci Hamiltonian by DFG:

Theorem (D.-Fillman-Gorodetski 2014)

We have $\sigma(H_V) = B$, and the Lebesgue measure of this set is zero. Moreover, we have $I(E) \geq 0$ for every $E \in \sigma(H_V)$.

Hausdorff Dimension of the Spectrum

The value of the invariant $I(E) = I(x_2(E), x_1(E), x_0(E))$ completely determines the local Hausdorff dimension of the spectrum at an energy $E \in \sigma(H_V)$.

Theorem (D.-Fillman-Gorodetski 2014)

There is a continuous map $D : [0, \infty) \to (0, 1]$ *with the following properties:*

(i) $\dim_{\mathrm{loc}}(\sigma(H_V), E) = D(I(E))$ *for every* $E \in \sigma(H_V)$.

(ii) We have $D(0) = 1$ *and* $1 - D(I) \asymp \sqrt{I}$ *as* $I \downarrow 0$.

(iii) We have

$$\lim_{I \to \infty} D(I) \cdot \log I = 2 \log(1 + \sqrt{2})$$

(iv) D *is real analytic in* $(0, \infty)$.

Hausdorff Dimension of the Spectrum

Remarks. (a) It follows immediately that the global Hausdorff dimension of the spectrum is always strictly positive.
(b) It was shown in a follow-up work by Jake Fillman and May Mei (AHP 2018) that the local Hausdorff dimension tends to one in both the weak-coupling limit and the high-energy limit. Thus, the global Hausdorff dimension of the spectrum is in fact equal to one.
(c) In the Kronig-Penney model, where the local bump functions are replaced by local point interactions, the local Hausdorff dimension of the spectrum can be equal to one for a sequence of energies tending to infinity. This can be seen via explicit calculations carried out in the DFG paper. For example, if $\ell_a = \ell_b = 1$ and $f_a(x) = \lambda\delta(x)$, $f_b(x) = 0$, we have $I(E) = \frac{\lambda^2}{4E} \sin^2 \sqrt{E}$. This observation explains the occurrence of so-called *pseudo bands* in the spectrum that had been pointed out earlier in the physics literature.

Zero Hausdorff Dimension via Limit Periodicity

A potential $V : \mathbb{R} \to \mathbb{R}$ is called *limit-periodic* if it is a uniform limit of continuous periodic functions on $\mathbb{R}$.

Denote the set of limit-periodic potentials by LP. It is naturally equipped with the L^∞ norm.

Theorem (D.-Fillman-Lukic 2017)

There is a dense set $\mathcal{H} \subseteq \mathrm{LP}$ such that for all $V \in \mathcal{H}$ and all $\lambda > 0$, $\sigma(H_{\lambda V})$ has Hausdorff dimension zero.

This result also has a "discrete precursor": a 2009 paper by Avila.