Recent advances on mean-field spin glasses

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Joint work with
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  - In mathematics: quenched disorder + frustration

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  - In physics: spin + glass
  - In mathematics: quenched disorder + frustration
- Spin glass features appear in many real world problems:
  - Traveling salesman problem.
  - Hopfield neural network.
  - Spike detection and recovery problems.

Edwards-Anderson model

- Consider a finite graph (V, E) on $\mathbb{Z}^d$.
- Hamiltonian: For $\sigma \in \{-1, 1\}^V$,

$$H(\sigma) = \sum_{(i,j) \in E} g_{ij} \sigma_i \sigma_j,$$

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  where g_{ij} are i.i.d. $N(0, 1)$.
- Frustration appears when computing $\max H_N(\sigma)$.

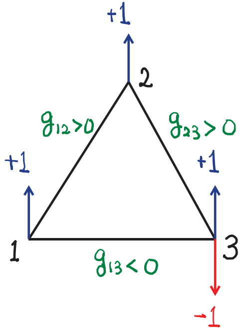

Figure: Frustration

Mean field approach: The Sherrington-Kirkpatrick model

- Hamiltonian:

$$H_N(\sigma) = \frac{1}{\sqrt{N}} \sum_{i,j=1}^{N} g_{ij}\sigma_i\sigma_j + h \sum_{i=1}^{N} \sigma_i$$

for $\sigma \in \{-1, +1\}^N$, where $g_{ij} \overset{i.i.d.}{\sim} N(0, 1)$.

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- Covariance Structure:

$$\mathbb{E}\Big(\frac{1}{\sqrt{N}} \sum_{i,j=1}^{N} g_{ij}\sigma_i^1\sigma_j^1\Big)\Big(\frac{1}{\sqrt{N}} \sum_{i,j=1}^{N} g_{ij}\sigma_i^2\sigma_j^2\Big) = N\big(R(\sigma^1,\sigma^2)\big)^2,$$

where

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Dean's problem

Assign N students into two dorms and avoid conflicts.

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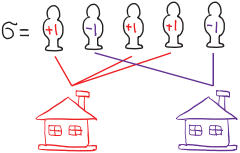

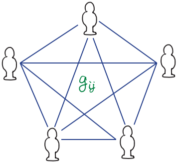

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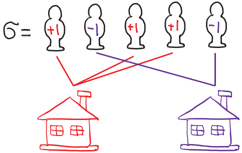

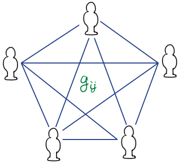

σ_i	σ_j	g_{ij}	$g_{ij}\sigma_i\sigma_j$
+1	+1	>0	>0
		<0	<0
+1	-1	>0	<0
		<0	>0
-1	-1	>0	>0
		<0	<0

Peace

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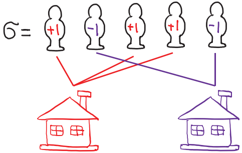

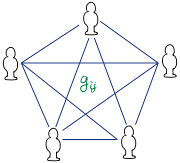

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+1	-1	>0	<0	
		<0	>0	Peace
-1	-1	>0	>0	Peace
		<0	<0	

Dean's problem: Find the optimizer of

$$\max_{\sigma \in \{-1,+1\}^N} \sum_{i,j=1}^{N} g_{ij}\sigma_i\sigma_j.$$

A soft approximation: Free energy

- For any $\beta = \frac{1}{T} > 0$ (inverse temperature), define the free energy

$$F_N(\beta) = \frac{1}{\beta N} \log \sum_{\sigma \in \{-1,+1\}^N} e^{\beta H_N(\sigma)}$$

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- Simple observation:

$$\max_{\sigma \in \{-1,+1\}^N} \frac{H_N(\sigma)}{N} \leq F_N(\beta) \leq \max_{\sigma \in \{-1,+1\}^N} \frac{H_N(\sigma)}{N} + \frac{\log 2}{\beta}$$

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- Physicists' replica method:

$$\lim_{N\to\infty} \frac{1}{N} \mathbb{E} \log Z_N = \lim_{N\to\infty} \lim_{n\downarrow 0} \frac{\mathbb{E} \log Z_N^n}{nN} \overset{?}{=} \lim_{n\downarrow 0} \lim_{N\to\infty} \frac{\log \mathbb{E} Z_N^n}{nN}$$

Theorem (Parisi formula)

- (Talagrand '06)

$$\lim_{N\to\infty} F_N(\beta) = \inf_{\alpha}\Big(\Phi_{\alpha,\beta}(0,h) - \frac{1}{2}\int_0^1 \beta\alpha(s)sds\Big),\ a.s.,$$

where for any CDF α *on* $[0,1]$,

$$\partial_s \Phi_{\alpha,\beta} = -\frac{1}{2}\Big(\partial_{xx}\Phi_{\alpha,\beta} + \beta\alpha(s)(\partial_x\Phi_{\alpha,\beta})^2\Big), \forall (s,x) \in [0,1) \times \mathbb{R}$$

with

$$\Phi_{\alpha,\beta}(1,x) = \frac{1}{\beta}\log\cosh(\beta x).$$

- (Guerra' 03) *Minimizer exists.*
- (Auffinger-C. '14) *Minimizer is unique.*

Denote this minimizer by α_β and call it the Parisi measure.

Significance of the Parisi measure

Three major predictions:

(1) α_β is the limiting distribution of the overlap:

$$R(\sigma^1,\sigma^2) \overset{d}{\Rightarrow} \alpha_\beta,$$

where σ^1,σ^2 are i.i.d. samplings from the Gibbs measure

$$G_N(\sigma) = \frac{e^{\beta H_N(\sigma)}}{\sum_{\sigma'} e^{\beta H_N(\sigma')}}.$$

(2) Phase Transition:

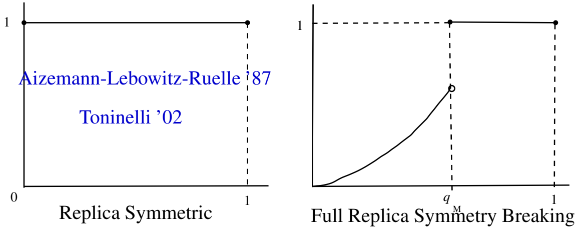
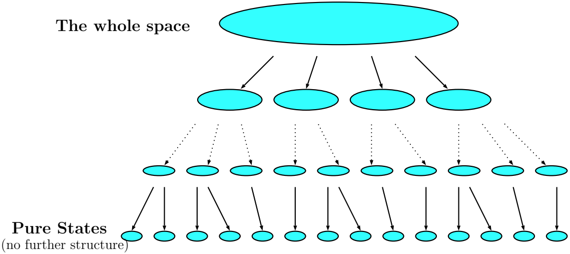


Figure: SK model with $h = 0$

(3) Ultrametricity: with probab.≈ 1, for i.i.d. $\sigma^1, \sigma^2, \sigma^3 \sim G_N$,

$$\|\sigma^1 - \sigma^2\| \leq \max\big(\|\sigma^1 - \sigma^3\|, \|\sigma^2 - \sigma^3\|\big) + o(1).$$



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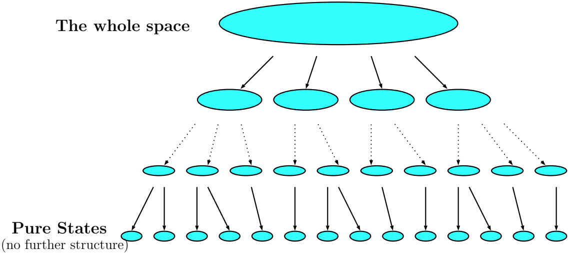

Panchenko '11: Ultrametricity holds for the SK model with a vanishing perturbation, but we do not know if it is still true without perturbation.

Theorem (Auffinger-C.-Zeng '17)

The cardinality of supp α_β *diverges as* $\beta \to \infty$.

As a consequence: If we add perturbation so that ultrametricity holds, then the total levels of the trees diverge as $\beta \uparrow \infty$.

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For any γ with $\gamma(s) = \mu([0,s])$ and $\int_0^1 \gamma(s)ds < \infty$, consider the PDE solution Ψ_γ,

$$\Psi_\gamma(1,x) = |x|,$$
$$\partial_s \Psi_\gamma = -\frac{1}{2}\Big(\partial_{xx}\Psi_\gamma + \gamma(s)(\partial_x \Psi_\gamma)^2\Big), \forall (s,x) \in [0,1) \times \mathbb{R}.$$

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Theorem

- *(Auffinger-C. '16) Parisi formula at zero temperature:*

$$\lim_{N\to\infty} \mathbb{E} \max_{\sigma\in\{-1,+1\}^N} \frac{H_N(\sigma)}{N} = \inf_\gamma \Big(\Psi_\gamma(0,h) - \frac{1}{2}\int_0^1 s\gamma(s)ds\Big)$$

- *(C.-Handschy-Lerman '16) Minimizer γ_P exists and is unique.*

Energy landscape: multiple peaks

Overlap $R(\sigma, \sigma') = \frac{1}{N} \sum_{i=1}^{N} \sigma_i \sigma'_i$.

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Theorem (Multiple peaks, C.-Handschy-Lerman '16)

Assume $h = 0$. For any $\varepsilon > 0$, there exists a constant $K > 0$ s.t. for any $N \geq 1$, with probability at least $1 - Ke^{-N/K}$, $\exists S_N \subset \{-1, +1\}^N$ such that

(i) $|S_N| \geq e^{N/K}$.

(ii) $\forall \sigma \in S_N$, $\left| \frac{H_N(\sigma)}{N} - \max_{\sigma' \in \Sigma_N} \frac{H_N(\sigma')}{N} \right| < \varepsilon$.

(iii) $\forall \sigma, \sigma' \in S_N$ *with* $\sigma \neq \sigma'$, $|R(\sigma, \sigma')| < \varepsilon$.

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- Chatterjee '09: $|S_N| \geq (\log N)^c$.
- Ding-Eldan-Zhai '14: $|S_N| \geq N^c$.

Pure p-spin model for $p \geq 3$: Overlap gap property

- Hamiltonian:

$$H_N(\sigma) = \frac{1}{N^{(p-1)/2}} \sum_{1 \leq i_1, \ldots, i_p \leq N} g_{i_1, \ldots, i_p} \sigma_{i_1} \cdots \sigma_{i_p}.$$

- (Overlap gap property) There exist $c, C > 0$ such that with overwhelming probability, any two near ground states σ^1 and σ^2 satisfy

$$|R(\sigma^1, \sigma^2)| \notin [c, C].$$

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- Results:
  - C.-Gamarnik-Rahman-Panchenko '17
  - Jagannath-Ben Arous '17

New challenges

Bipartite SK model: Let $N_1 = cN$ and $N_2 = (1-c)N$.

$$H_N(\sigma) = \frac{1}{\sqrt{N}} \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} g_{ij} \tau_i \rho_j$$

for $\sigma = (\tau, \rho) \in \{-1,+1\}^{N_1} \times \{-1,+1\}^{N_2}$. Note

$$\mathbb{E} H_N(\sigma) H_N(\sigma') = c(1-c) N R(\tau, \tau') R(\rho, \rho').$$

Questions:

- Free energy?
- Ground state energy?
- Energy landscape?

Thank you for your attention.