On foliations related to the center of mass in General Relativity

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Isolated system in General Relativity

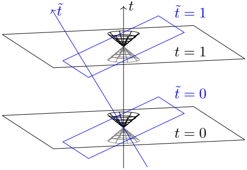
Consider initial data (M^3, g, K, μ, J) which are "optimally" asymptotically flat:

$$
\begin{aligned}
M^3 &\approx \mathbb{R}^3 \setminus \text{ball} \ni \vec{x} \\
g_{ij} &= \delta_{ij} + \mathcal{O}_2(r^{-\frac{1}{2}-\varepsilon}) \\
K_{ij} &= \qquad \mathcal{O}_1(r^{-\frac{3}{2}-\varepsilon}), \\
\mu, J &= \qquad \mathcal{O}_0(r^{-3-\varepsilon})
\end{aligned}
$$

for some $\varepsilon > 0$ and $r = |\vec{x}| \to \infty$.

Expectations of a notion of center of mass

- Transforms like a point particle in Special Relativity under change of observer:
  $\rightsquigarrow$ equivariant transformation behavior under asymptotic boosts



- Equivariant transformation under spatial translations and rotations.
- Point particle-like evolution under Einstein evolution equations:

$$\frac{d}{dt}(E\vec{z}) = \vec{P}$$

(ADM-energy E, ADM-momentum $\vec{P}$)

- Newtonian limit $c \to \infty$ of $\vec{z}(c)$ recovers Newtonian center of mass of $\vec{z}$ limiting Newtonian isolated system

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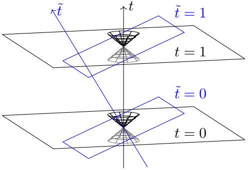

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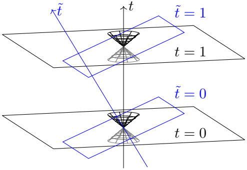

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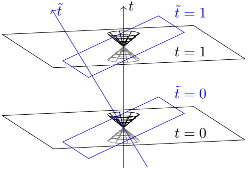

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Status quo

Different definitions of center of mass in the literature:

- Definition via Hamiltonian systems:
  Regge–Teitelboim '74, Beig–Ó Murchadha '87.
  $\leadsto$ does not transform equivariantly and does not converge in general
- Asymptotic foliation definition by Huisken–Yau '96.
  $\leadsto$ see below
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Excursion: Isolated systems in Newtonian Gravity

Center of mass $\vec{z}_N \in \mathbb{R}^3$ of a mass density ρ and mass $m_N = \int_{\mathbb{R}^3} \rho \, dV \neq 0$:

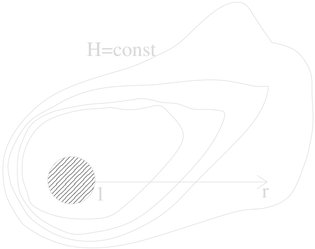
$$\vec{z}_N = \frac{1}{m_N} \int_{\mathbb{R}^3} \rho \, \vec{x} \, dV.$$

Can be reformulated: U Newtonian potential with $U \to 0$ as $r \to \infty$:

$$\triangle U = 4\pi\rho.$$

If $m_N \neq 0$: equipotential sets Σ_U foliate neighborhood of infinity. Recover $\vec{z}_N$ from

$$\vec{z}_N = \lim_{U \to 0} \int_{\Sigma_U} \vec{x} \, dA.$$



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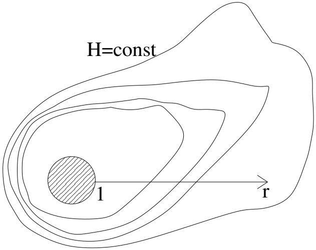
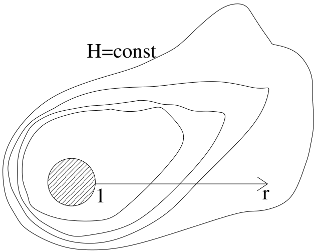


Huisken–Yau definition of center of mass I

Theorem (Huisken–Yau '96; *abstract CoM*)

Let (M^3, g) be an asymptotically spherically symmetric Riemannian manifold of masse $m > 0$. There exists an (almost) unique foliation of a neighborhood of infinity by stable spheres Σ_H of constant mean curvature H (CMC).



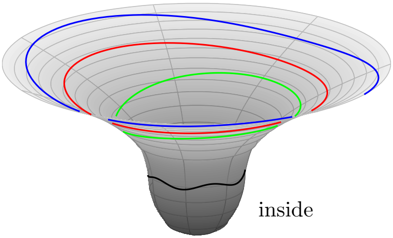
- Asymptotic condition: $g_{ij} = \left(1 + \frac{m}{2r}\right)^4 \delta_{ij} + \mathcal{O}_4(\frac{1}{r^2})$.
- Generalizations: Ye, Metzger, Metzger–Eichmair, Huang, Nerz, ...

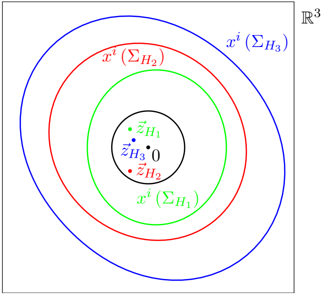
Huisken–Yau definition of center of mass II

Theorem (Huisken–Yau '96; *coordinate CoM*)

Euclidean center $\vec{z}_H$ of Σ_H and center of mass $\vec{z}_{HY}$:

$$\vec{z}_H := \fint_{\vec{x}\,(\Sigma_H)} \vec{x}\, dA, \quad \vec{z}_{HY} := \lim_{H\to 0} \vec{z}_H .$$





However:

Theorem (C.–Nerz '14)

Der center of mass $\vec{z}_{HY} := \lim_{H\to 0} \vec{z}_H$ does not always converge under the assumptions of Huisken–Yau.

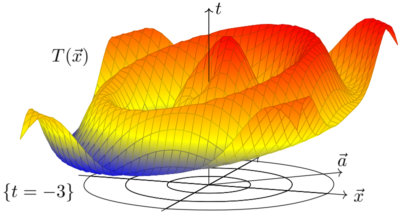


Figure: Logarithmic plot.

Explicit counterexample: graphical timeslice in Schwarzschild spacetime:

$$T(\vec{x}) = \frac{\vec{a}\cdot\vec{x}}{r} + \sin(\ln r),$$

$$\vec{a} \in \mathbb{R}^3, \vec{a} \neq 0$$

- Reason: $\mathrm{R}\,\vec{x} \notin L^1$ in general, R scalar curvature of g.
- Same phenomenon in Newtonian setting by changing coordinates asymptotically if $\rho\,\vec{x} \notin L^1$.

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Theorem (C.–Nerz '14)

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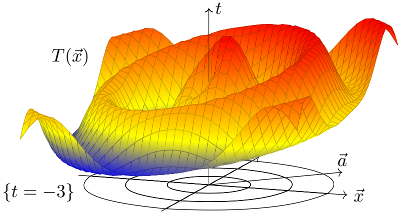

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New development

Theorem (C.–Sakovich '18)

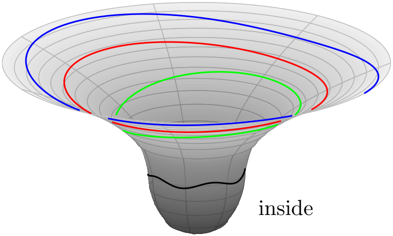
Let (M^3, g, K, μ, J) *be initial data. Under optimal asymptotic flatness conditions and if the ADM-energy* $E \neq 0$*, there exists a unique foliation of a neighborhood of infinity by stable spheres* $\Sigma_{\mathcal{H}}$ *of constant spacetime mean curvature* $\mathcal{H} = \sqrt{\mathfrak{g}(\vec{\mathcal{H}}, \vec{\mathcal{H}})}$ *(STCMC).*

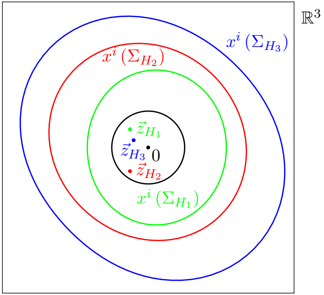
Assuming $\mu\,\vec{x} \in L^1$*, the euclidean center* $\vec{z}_{\mathcal{H}}$ *of* $\Sigma_{\mathcal{H}}$ *and the center of mass* $\vec{z}$ *satisfies*[a]*:*

$$\vec{z}_{\mathcal{H}} := \fint_{x^i(\Sigma_{\mathcal{H}})} \vec{x}\, dA, \quad \vec{z} := \lim_{\mathcal{H}\to 0} \vec{z}_{\mathcal{H}}.$$

[a] Under a weak additional decay assumption on K which seem technical.

Coordinate STCMC-center of mass





New development. . .

Theorem (C.–Sakovich '18)

- *The STCMC-center of mass $\vec{z}$ transforms equivariantly under the asymptotic Poincaré group (in coordinates), i.e. under boosts and spatial translations and rotations, as well as*
- *point particle-like evolution under the Einstein evolution equations via*

$$\frac{d}{dt}(E\vec{z}) = \vec{P}.$$

- *The counterexample from [C.–Nerz '14] has a well-defined STCMC-center of mass $\vec{z} = \vec{0}$.*

Proof: Method of continuity, implicit function theorem in Sobolev spaces, spectral analysis of new STCMC-stability operator, results by Nerz '15, '16, . . .

New development. . .

- Have explicit formula for difference between $\vec{z}_{HY}$ und new $\vec{z}$ via BÓM–RT-formula (Huang, Nerz, . . .).
- Agrees with Chen–Wang–Yau center of mass if initial data are asymptotically harmonic.

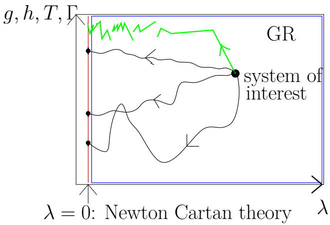
- Work in progress with Metzger: The extra weak additional decay assumption on K is not necessary but can be replaced by choosing suitable center of mass coordinates.

Open question: Newtonian limit of center of mass...

Theorem (C. '11)

Along each c-dependant family of static isolated systems that has a Newtonian limit as $c \to \infty$, one finds that

$$\vec{z}_{HY}(c) = \vec{z}_{BÓM\text{–}RT}(c) = \vec{z}_{PN}(c) \to \vec{z}_{N}.$$



Proof: Ehlers' frame theory, differential geometry modelling, Kelvin transformation, weighted Sobolev space analysis, faster fall-off trick [C. '11], localization of mass and center of mass via pseudo-Newtonian gravity.