Eigenvector Correlations for the Ginibre Ensemble

Nick Crawford; The Technion

July 24, 2018

joint with Ron Rosenthal

The Ginibre Ensemble

Let $(m_{ij})_{i,j\in\mathbb{N}}$ be i.i.d. $\mathcal{N}_{\mathbb{C}}(0,1/N)$ variables. We consider the matrix

$$M_N := (m_{ij})_{i,j\leq N}$$

acting on $\mathbb{C}^N$.

- What are the statistical properties of this matrix ensemble?
- **Eigenvalues:** Almost surely, M_N is diagonalizable. With respect to Lebesgue measure $\prod_{i=1}^N \mathrm{d}^2\lambda_i$, the density is

$$\frac{\mathrm{d}\mathbb{P}(\underline{\lambda})}{\prod_{i=1}^N \mathrm{d}^2\lambda_i} = \frac{1}{Z_N}\prod_{i<j\leq N}|\lambda_i-\lambda_j|^2\prod_{i\leq N}\exp(-N|\lambda_i|^2)$$

- Asymptotic density of states is uniform over unit disc $\mathbf{D}_1\subset\mathbb{C}$.

- Quite a bit is known about asymptotic behavior of eigenvalues in this ensemble and various generalizations.
  Ginibre '65; Girko '84, '94; Bai 97; Tao, Vu '08, '10; Götze, Tikhomirov '10; Bourgade, Yau, Yin '14a, '14b; Yin '14; Alt, Erdös, Krueger '18.
- Important fact: $\|(M_N^* M_N)^{-1/2}\|_\infty \sim N$, where as eigenvalue spacing is $N^{-1/2}$.
- In spite of this, much less is understood regarding the eigenvector geometry. Note however Rudelson, Vershynin '15.

- Quite a bit is known about asymptotic behavior of eigenvalues in this ensemble and various generalizations.
  Ginibre '65; Girko '84, '94; Bai 97; Tao, Vu '08, '10; Götze, Tikhomirov '10; Bourgade, Yau, Yin '14a, '14b; Yin '14; Alt, Erdös, Krueger '18.
- Important fact: $\|(M_N^* M_N)^{-1/2}\|_\infty \sim N$, where as eigenvalue spacing is $N^{-1/2}$.
- In spite of this, much less is understood regarding the eigenvector geometry. Note however Rudelson, Vershynin '15.
- **Which Eigenvectors?** Given the eigenvalues $(\lambda_i)_{i=1}^N$, associate TWO bases:

$$\begin{aligned} &\text{Column vectors:} \quad M_N \cdot r_i = \lambda_i r_i, \\ &\text{Row vectors:} \quad \ell_i \cdot M_N = \lambda_i \ell_i, \\ &\text{Normalization:} \quad \ell_i \cdot r_j = \delta_{i,j}. \end{aligned}$$

- Then with $Q_i = r_i \otimes \ell_i$,

$$M_N = \sum_i \lambda_i Q_i.$$

Statistics of the Q_i's

Chalker-Mehlig '98:
Let $M_N(0), M_N(1)$ be independent copies of M_N and set

$$M_N(\theta) = \cos(\theta) M_N(0) + \sin(\theta) M_N(1).$$

Then (at $\theta = 0$), eigenvalue trajectories $(\lambda_i(\theta))_{i \leq N}$ satisfy

$$\mathbb{E}[\partial_\theta \lambda_i \partial_\theta \overline{\lambda_j} | \lambda_i(0), \lambda_j(0)] = \frac{1}{N} \mathbb{E}[\mathrm{Tr}(Q_i Q_j^*) | \lambda_i(0), \lambda_j(0)],$$

$$\frac{1}{N} \mathbb{E}[\mathrm{Tr}(Q_i \cdot Q_j^*) | \lambda_i(0), \lambda_j(0)] \sim \begin{cases} 1 - |\lambda_i|^2 \text{ if } i = j, \\ \frac{1}{N^2} \frac{1 - \lambda_i \overline{\lambda_j}}{|\lambda_i - \lambda_j|^4} \text{ if } i \neq j, \end{cases}$$

for typical eigenvalues.

Subsequent Work

- "Polish Group": Burda, Nowak et al. ('99), Burda, Grela, Nowak et al. ('14), Belinshi, Nowak, et al. ('16);
- Starr, Walters ('14). Corrections to **CM-'98** at $\partial \mathbf{D}_1$.
- Fyodorov ('17); Bourgade, Dubach ('18). Conditional on λ_i in bulk,

$$\frac{1}{N(1 - |\lambda_i^2|)} \mathrm{Tr}[Q_i Q_i^*]$$

scales to $1/\Gamma(2)$.

Higher Order Correlations

- Given $A \subset \mathbb{N}$, let $\mathcal{S}_A$ be the permutation group on A.
- If $\mathcal{L} \in \mathcal{S}_A$ is a cycle let

$$\hat{\rho}(\mathcal{L}) = N^{|A|-1} \mathrm{Tr}\left[\prod_{j \in A} Q^*_{2j-1} Q_{2j}\right].$$

with cycle order imposed.

- For $\sigma \in \mathcal{S}_k$ set

$$\hat{\rho}(\sigma) = \prod_{\mathcal{L} \text{ cycles of } \sigma} \hat{\rho}(\mathcal{L}).$$

Finally given $\mathbf{u}, \mathbf{v} \in \mathbf{D^k}$,

$$\rho_N(\sigma) = \mathbb{E}[\hat{\rho}(\sigma) | \lambda_{2j-1} = u_j, \lambda_{2j} = v_j \text{ for } j \in \{1, \dots k\}].$$

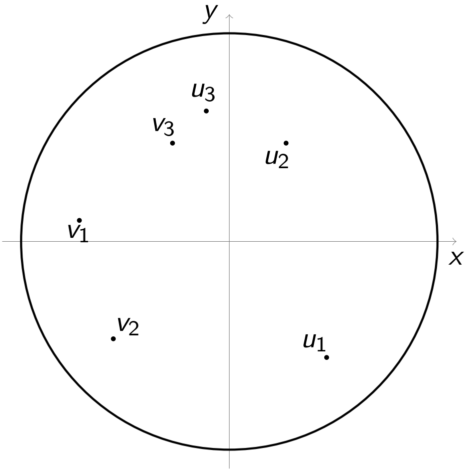


Figure: Schematic for $\mathrm{Tr}[Q_1^* Q_2 Q_3^* Q_4 Q_5^* Q_6]$ and $\rho_N(123; \mathbf{u}, \mathbf{v})$

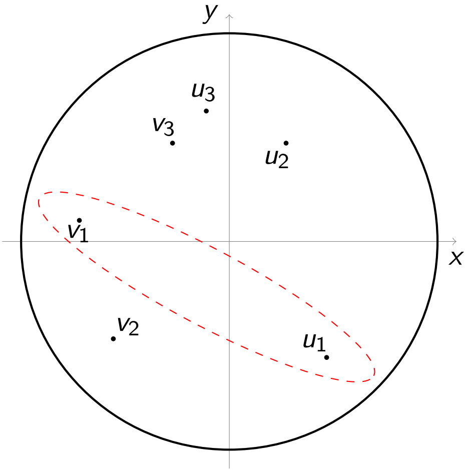


Figure: Schematic for $\mathrm{Tr}[Q_1^* Q_2 Q_3^* Q_4 Q_5^* Q_6]$ and $\rho_N(123; \mathbf{u}, \mathbf{v})$

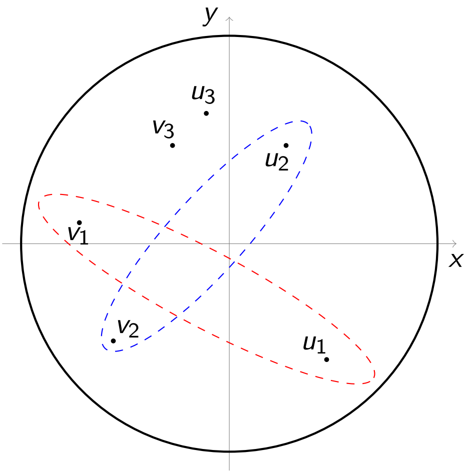


Figure: Schematic for $\mathrm{Tr}[Q_1^* Q_2 Q_3^* Q_4 Q_5^* Q_6]$ and $\rho_N(123; \mathbf{u}, \mathbf{v})$

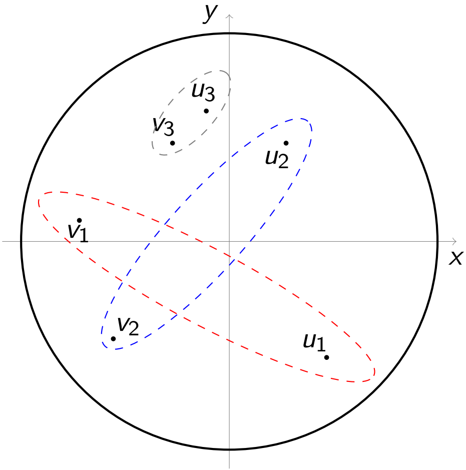


Figure: Schematic for $\mathrm{Tr}[Q_1^* Q_2 Q_3^* Q_4 Q_5^* Q_6]$ and $\rho_N(123; \mathbf{u}, \mathbf{v})$

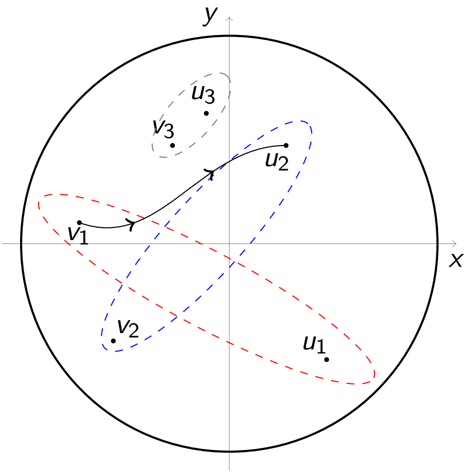


Figure: Schematic for $\mathrm{Tr}[Q_1^* Q_2 Q_3^* Q_4 Q_5^* Q_6]$ and $\rho_N(123; \mathbf{u}, \mathbf{v})$

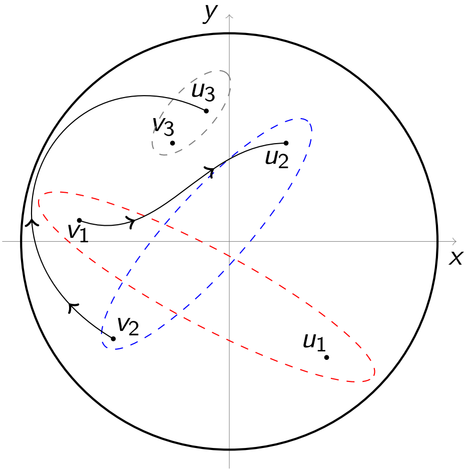


Figure: Schematic for $\mathrm{Tr}[Q_1^* Q_2 Q_3^* Q_4 Q_5^* Q_6]$ and $\rho_N(123; \mathbf{u}, \mathbf{v})$

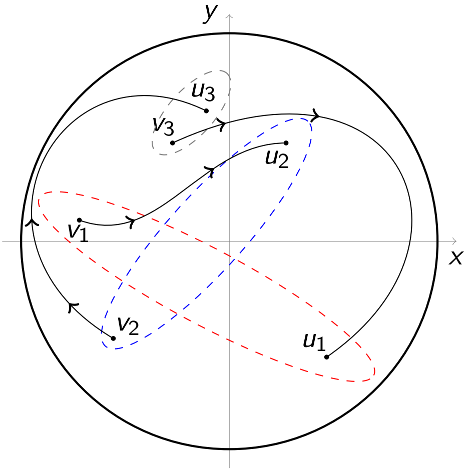


Figure: Schematic for $\mathrm{Tr}[Q_1^* Q_2 Q_3^* Q_4 Q_5^* Q_6]$ and $\rho_N(123; \mathbf{u}, \mathbf{v})$

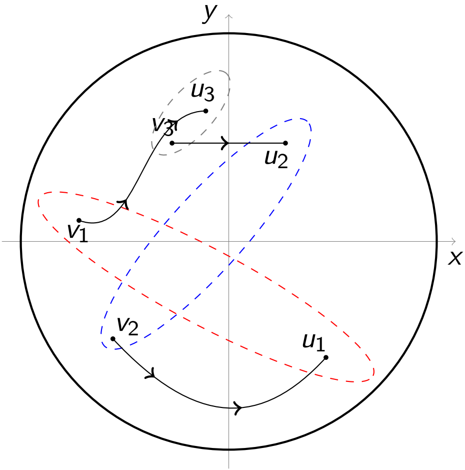


Figure: Schematic for $\mathrm{Tr}[Q_1^* Q_2 Q_3^* Q_4 Q_5^* Q_6]$ and $\rho_N(132; \mathbf{u}, \mathbf{v})$

Macroscopic Factorization

For $\mathbf{u}, \mathbf{v} \in \mathbf{D}_1^k$, define

$$\mathrm{Dist}(\mathbf{u},\mathbf{v}) := \min_{\alpha,\beta\in[k]}\{|u_\alpha - v_\beta|\} \wedge \min_{\alpha,\beta\in[k],\ \alpha\neq\beta}\{|u_\alpha - u_\beta|, |v_\alpha - v_\beta|\} \wedge \min_{\alpha\in[k]}\{1-|u_\alpha|, 1-|v_\alpha|\}. \quad (1)$$

Theorem

For every $\sigma \in \mathcal{S}_k$ and every $\mathbf{u}, \mathbf{v} \in \mathbf{D}_1^k$ such that $\mathrm{Dist}(\mathbf{u},\mathbf{v}) > 0$, the limit

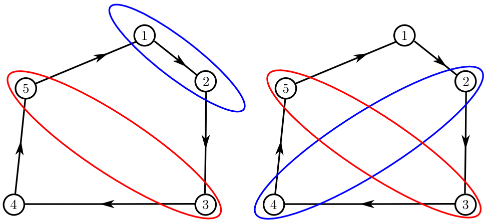
$$\rho(\sigma; \mathbf{u}, \mathbf{v}) := \lim_{N\to\infty} \rho_N(\sigma; \mathbf{u}, \mathbf{v})$$

exists. If $\sigma = \{\mathcal{L}_j\}_{j=1}^{|\sigma|}$ with $\mathcal{L}_j$ the cycles of σ

$$\rho(\sigma; \mathbf{u}, \mathbf{v}) = \prod_{j=1}^{|\sigma|} \rho(\mathcal{L}_j; \mathbf{u}|_{\mathcal{V}(\mathcal{L}_j)}, \mathbf{v}|_{\mathcal{V}(\mathcal{L}_j)}).$$

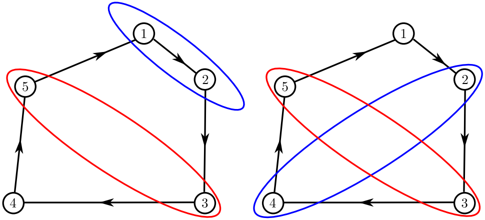
From now on $C_k = (12 \cdots k)$.

- Let π_1, π_2 be cyclic permutations on disjoint subsets A, B of $[k]$. We say they are *crossing* if there exists $\alpha \in A$ and $\beta \in B$ such that $(\alpha, \pi_1(\alpha), \beta, \pi_2(\beta))$ is not the ordering of these vertices in C_k. Otherwise, we call them *noncrossing*.



From now on $C_k = (12 \cdots k)$.

- Let π_1, π_2 be cyclic permutations on disjoint subsets A, B of $[k]$. We say they are *crossing* if there exists $\alpha \in A$ and $\beta \in B$ such that $(\alpha, \pi_1(\alpha), \beta, \pi_2(\beta))$ is not the ordering of these vertices in C_k. Otherwise, we call them *noncrossing*.



- Say that $\pi \in \mathcal{S}_k$ is *noncrossing* if its cycles are pair-wise noncrossing. Denote this property by $\pi \trianglelefteq C_k$.
- Let $\mathrm{V}_k(\mathbf{v})$ be the Vandermonde determinant $\prod_{\alpha,\beta \in [k],\, \alpha<\beta} (v_\beta - v_\alpha)$.

Correlation Structure of a Cycle

Theorem

There are two families of polynomials $(\mathfrak{R}_\pi, \mathfrak{L}_\pi)_{\pi \in \mathcal{S}}$ *in* $\mathbf{u}, \mathbf{v} \in \mathbb{C}^k \times \mathbb{C}^k$, *homogeneous of degree of degree* $\binom{k-1}{2}$, *so that*

$$\rho(C_k; \mathbf{u}, \mathbf{v}) = \sum_{\substack{\pi \trianglelefteq C_k \\ \mathcal{V}(\pi) = [k]}} \frac{\mathfrak{R}_\pi(\overline{\mathbf{u}}, \mathbf{v}) \mathfrak{L}_{\sigma \circ \pi^{-1}}(\overline{\mathbf{u}}, \pi^{-1}(\mathbf{v}))}{\mathrm{V}_k(\overline{\mathbf{u}})^2 \mathrm{V}_k(\mathbf{v})^2} \prod_{\alpha \in \mathcal{V}(\pi)} \rho_2(u_\alpha, v_{\pi^{-1}(\alpha)}),$$

where

$$\rho_2(z, w) = \frac{1 - \overline{z} w}{|z - w|^4}$$

Example:

$$\rho_4(\nu_1, \nu_2, \nu_3, \nu_4) = \frac{1}{(\nu_2 - \nu_4)^2 (\overline{\nu_1} - \overline{\nu_3})^2} \left[\rho_2(\nu_1, \nu_2) \rho_2(\nu_3, \nu_4) - \rho_2(\nu_1, \nu_4) \rho_2(\nu_3, \nu_2) \right].$$

Origin of the Polynomials

- For $\sigma, \tau \in \mathcal{S}_k$ say $\sigma \preceq \tau$ if $\mathcal{V}(\sigma) \subset \mathcal{V}(\tau)$, every cycle of σ is a subcycle of τ and all but at most one of the cycles of τ are also cycles in σ.
- Let

$$h(u, v) = \frac{1}{\pi} \int_{\mathbf{D}_1} \frac{1}{(\overline{\nu} - \overline{u})(\nu - v)} \mathrm{d}^2\nu = \log\left(\frac{1 - \overline{u}v}{|u - v|^2}\right).$$

and note that $\partial_u \partial_{\overline{v}} h(u, v) = \rho_2(u, v) = \frac{1 - u\bar{v}}{|u - v|^4}$.

Let

$$h(\pi) = \sum_{\alpha=1}^{k} h(u_\alpha, v_{\pi^{-1}(\alpha)}).$$

Theorem

There is a matrix $\mathfrak{N} : \mathbb{C}^{\mathcal{S}_k} \to \mathbb{C}^{\mathcal{S}_k}$*, parametrized by* $\mathbf{u}, \mathbf{v} \in \mathbf{D}_1^k$*, and upper triangular w.r.t.* $\preceq$ *so that:*

1. $\rho(\sigma) = \partial_{\mathbf{u}} \partial_{\overline{\mathbf{v}}} e^{\mathfrak{N}}(Id, \sigma)$,
2. *The eigenvalues of* $\mathfrak{N}$ *are* $h(\pi)'s$ *and the eigenvector components are rational in* $\overline{\mathbf{u}}, \mathbf{v}$ *(!).*