Multiplicative chaos in random matrix theory and related fields

Christian Webb

Aalto University, Finland

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Theorem (Charlier 2017)

Let $x_1, \ldots, x_k \in (-1,1)$ be fixed and distinct. Then

$$\lim_{N\to\infty} \mathbb{E} \prod_{j=1}^{k} \frac{e^{\gamma V_N(x_j)}}{\mathbb{E}e^{\gamma V_N(x_j)}} = \prod_{1\leq p<q\leq k} \left| \frac{1 - x_p x_q + \sqrt{1-x_p^2}\sqrt{1-x_q^2}}{x_p - x_q} \right|^{\frac{\gamma^2}{2\pi^2}}$$

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- **Is there a process with such moments? Does $\frac{e^{\gamma V_N(x)}}{\mathbb{E}e^{\gamma V_N(x)}}$ converge to it? What would this say about the GUE?**

The limiting process – heuristics

- For $(Y_k)_{k=1}^{\infty}$ i.i.d. standard Gaussians and $(U_j)_{j=0}^{\infty}$ Chebyshev polynomials of the second kind, let (formally):

$$X(x) = \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{Y_k}{\sqrt{k}} U_{k-1}(x) \sqrt{1 - x^2}.$$

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- For $\mu_\gamma(x) = e^{\gamma X(x) - \frac{\gamma^2}{2} \mathbb{E} X(x)^2}$ (formally)

$$\mathbb{E} \prod_{j=1}^{k} \mu_\gamma(x_j) = \prod_{1 \le p < q \le k} \left| \frac{1 - x_p x_q + \sqrt{1-x_p^2}\sqrt{1-x_q^2}}{x_p - x_q} \right|^{\frac{\gamma^2}{2\pi^2}}.$$

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- For each x, the sum defining $X(x)$ diverges almost surely and $\mathbb{E}X(x)^2=\infty$. What does μ_γ mean? ☹

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- **This procedure defines random measures/distributions. These are the objects we are after – correlation kernels agree with the limiting GUE-moments.**

Real Gaussian multiplicative chaos – the general picture

- A centered **log-correlated Gaussian field** $G(x)$ is (formally) a Gaussian process on $\mathbb{R}^d$ with covariance

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Theorem (Kahane 1985,...)

For nice enough $C(x,y)$, as $N \to \infty$:

- $e^{\gamma G_N(x) - \frac{\gamma^2}{2}\mathbb{E}G_N(x)^2}dx$ *converges to a non-trivial random measure M_γ for $-\sqrt{2d} < \gamma < \sqrt{2d}$. For $|\gamma| \geq \sqrt{2d}$, the limit is zero.*
- *For $|\gamma| < \sqrt{2d}$, M_γ lives on the random set of points (of dimension $d - \frac{\gamma^2}{2}$)*

$$\left\{ x \in \mathbb{R}^d : \lim_{N\to\infty} \frac{G_N(x)}{\mathbb{E}G_N(x)^2} = \gamma \right\}.$$

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- Interpretation: GMC$\to$ level sets. $\max_x G_N(x) \sim \sqrt{2d}\,\mathbb{E}G_N(x)^2$.

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- Has also been used in some models of mathematical finance.

The GUE e.v. counting function and GMC

Theorem (Claeys, Fahs, Lambert, W 2018)

Let $\gamma \in (-2\pi, 2\pi)$ *and* $f \in C_c((-1,1))$. *Then as* $N \to \infty$

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Corollary (Claeys, Fahs, Lambert, W 2018)

For any $\epsilon, \delta > 0$ *fixed,* $\lambda_1 \leq ... \leq \lambda_N$ *as before, and* ρ_k *the classical locations of the eigenvalues*:

$$\lim_{N\to\infty} \mathbb{P}\left(\frac{1}{\pi} - \epsilon \leq \sup_{\delta N \leq k \leq (1-\delta)N} \frac{N \frac{2}{\pi} \sqrt{1-\rho_k^2}}{\log N} |\lambda_k - \rho_k| \leq \frac{1}{\pi} + \epsilon \right) = 1.$$

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Let ω be uniformly distributed on $[0,1]$ and $x, y \in \mathbb{R}$ be fixed. As $T \to \infty$

$$\mathbb{E}\zeta\left(\tfrac{1}{2} + ix + i\omega T\right)\overline{\zeta\left(\tfrac{1}{2} + iy + i\omega T\right)}$$
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- Does $\lim_{T\to\infty} \zeta(\frac{1}{2} + ix + i\omega T)$ exist? ...

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Theorem (Saksman, W (2016))

- *For any* $f \in C_c^\infty(\mathbb{R}, \mathbb{C})$*,*

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- (Stronger results) conjectured by Fyodorov and Keating.

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Multiplicative chaos and the Riemann zeta

Theorem (Saksman, W (2016))

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- Proof philosophy similar to GUE. Methods fairly basic number theory.
- Geometric interpretation? Interesting results about $\max \mathrm{Re}/\mathrm{Im} \log \zeta(\frac{1}{2} + ix + i\omega T)$ exist: see Najnudel; Arguin et al.

The critical Ising model

- Let U be a bounded simply connected domain in $\mathbb{C}$ and U_δ a lattice approximation of U of mesh $\delta > 0$.

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Let ψ be any conformal bijection from U to the upper half plane and $\mathcal{C}$ a suitable constant. Then for $x_1, \ldots, x_k \in U$ fixed and distinct,

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- If σ_δ and $\widetilde{\sigma}_\delta$ are independent copies, does $x \mapsto \delta^{-1/4} \sigma_\delta(x) \widetilde{\sigma}_\delta(x)$ converge to some process (known that $\delta^{-1/8} \sigma_\delta(x)$ does)? ...

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Theorem (Junnila, Saksman, W 2018)

Let σ_δ and $\widetilde{\sigma}_\delta$ be independent copies of the Ising spin field. Then for any $f \in C_c^\infty(U)$, as $\delta \to 0$

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The GFF and cos(GFF): images