Lieb-Robinson bounds, Arveson spectrum and Haag-Ruelle scattering theory for gapped quantum spin systems

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joint work with Sven Bachmann[2] and Pieter Naaijkens[3]

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Introduction

Our observation: For a class of gapped quantum spin systems

- satisfying Lieb-Robinson bounds,
- admitting single-particle states

Haag-Ruelle scattering theory can be developed in a natural, model independent manner.

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Comparison with the literature

1. Haag-Ruelle scattering theory for Euclidean lattice quantum field theories.

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2. Scattering theory for quantum spin systems relying on properties of concrete Hamiltonians.

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Outline

1. Scattering in Quantum Mechanics
2. Scattering in QFT and spin systems
3. The problem of asymptotic completeness
4. Conclusions and outlook

Scattering in Quantum Mechanics

1. Hilbert space: $\mathcal{H} := L^2(\mathbb{R}^3, dx)$
2. Hamiltonian: $H = -\frac{1}{2}\Delta + V(x)$
3. Schrödinger equation: $\mathrm{i}\partial_t \Psi_t = H\Psi_t$
4. Time evolution: $\Psi_t := \mathrm{e}^{-\mathrm{i}tH}\Psi_{t=0}$

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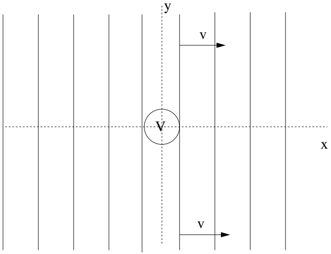

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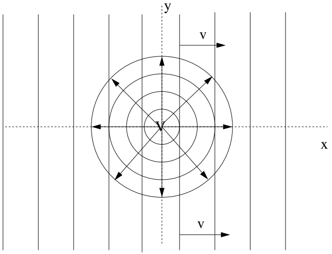

Scattering in Quantum Mechanics

1. There are states $\Psi^{\rm out} \in \mathcal{H}$ of the particle in potential V which for large times evolve like states of the free particle.
2. For any such $\Psi^{\rm out}$ there exists $\Psi \in \mathcal{H}$ s.t.
3. Def: $\Psi^{\rm out} := \lim_{t\to\infty} \mathrm{e}^{\mathrm{i}tH}\mathrm{e}^{-\mathrm{i}tH_0}\Psi$ is the scattering state.
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Cook's method

1. Let $\Psi_t := \mathrm{e}^{\mathrm{i}tH}\mathrm{e}^{-\mathrm{i}tH_0}\Psi$.
2. Suppose we can show
$$\|\partial_t \Psi_t\| = \|\mathrm{e}^{\mathrm{i}tH} V \mathrm{e}^{-\mathrm{i}tH_0}\Psi\| \in L^1(\mathbb{R}, dt).$$
3. Then $\lim_{t\to\infty} \Psi_t = \int_{t_0}^{\infty} (\partial_\tau \Psi_\tau) d\tau + \Psi_{t_0}$ exists.

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Framework for QFT and spin systems

1. Γ - the abelian group of space translations ($\mathbb{R}^d$ or $\mathbb{Z}^d$).
2. $\widehat{\Gamma}$ - Pontryagin dual of Γ ($\mathbb{R}^d$ or S_1^d).
3. $(\mathfrak{A}, \tau)$ - C^*-dynamical system with $\mathbb{R} \times \Gamma \ni (t, x) \mapsto \tau_{(t,x)}$.
4. $\mathfrak{B} \subset \mathfrak{A}$ -almost-local operators: $\|[B_1, \tau_{(s,vs)}(B_2)]\| = O(|s|^{-\infty})$.
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   Lieb-Robinson bounds:

$$\|[\tau_t(A), B]\| \leq C_{A,B} \mathrm{e}^{\lambda(v_{\mathrm{LR}} t - d(A,B))}, \quad A, B \in \mathfrak{A} \text{ local.}$$

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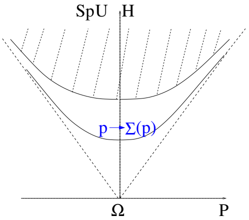

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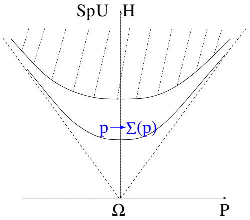
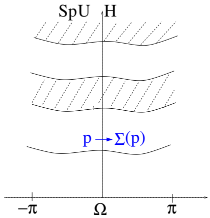




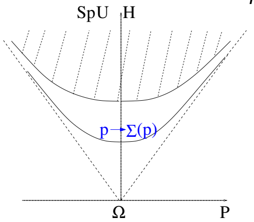
Examples

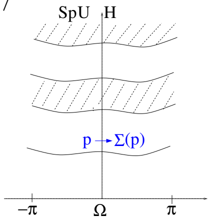
1. QFT: $\lambda\phi^4$ theory for 1 and 2 space dimensions:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4$$

2. Spin systems: Ising model in transverse magnetic field for any space dimension:

$$H = -\frac{1}{2}\sum_i(\sigma_i^{(z)} - 1) - \varepsilon\sum_{\langle i,j\rangle}\sigma_i^{(x)}\sigma_j^{(x)}$$





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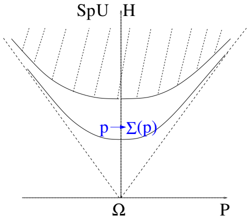

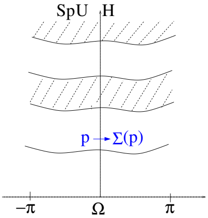

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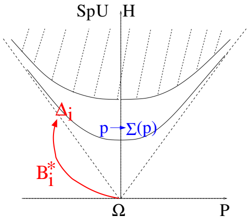

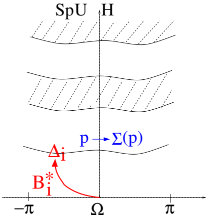

Let $B_1^*, \dots, B_n^* \in \mathfrak{B}$ be s.t. $B_i^*\Omega \in \mathbb{1}_U(\Delta_i)\mathcal{H}$ are single-particle states.

Arveson spectrum

Let $(\mathfrak{A}, \tau)$ be a C^*-dynamical system.

Definition

The Arveson spectrum of $A \in \mathfrak{A}$ is the support of the (inverse) Fourier transform of $\mathbb{R} \times \Gamma \ni (t, x) \mapsto \tau_{(t,x)}(A)$.

It is denoted $\mathrm{Sp}_A \tau$.

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Fact 1: (Energy-momentum transfer relation)

1. Let $\tau_{(t,x)}(A) = U(t,x)AU(t,x)^*$ for $A \in \mathfrak{A}$.
2. Let $\mathbb{1}_U(\,\cdot\,)$ denote the spectral measure of U.

Then

$$A\,\mathbb{1}_U(\Delta)\mathcal{H} \subset \mathbb{1}_U(\overline{\Delta + \mathrm{Sp}_A \tau})\mathcal{H}, \quad A \in \mathfrak{A}.$$

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Fact 2: For any compact Δ there are plenty almost-local operators A with $\mathrm{Sp}_A \tau \subset \Delta$.

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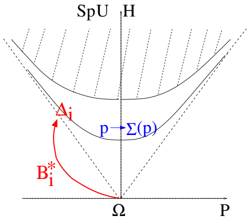

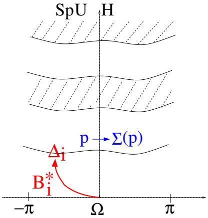

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Theorem (Haag-Ruelle 62, Bachmann-Naaijkens-W.D.)

The following limits exist and are called scattering states

$$\Psi^{\mathrm{out}} := \lim_{t\to\infty} B^*_{1,t}(g_{1,t})\ldots B^*_{n,t}(g_{n,t})\Omega, \qquad \textit{where}$$

$B^*_t(g_t) := \int_\Gamma d\mu(x)\tau_{(t,x)}(B^*)g_t(x)$, $g_t(x) := \int_{\hat{\Gamma}} dp\, e^{-i\Sigma(p)t+ipx}\hat{g}(p)$

and velocity supports $V(g_i) := \{\, \nabla\Sigma(p) \,|\, p\in \mathrm{supp}\,\hat{g}_i \,\}$ *are disjoint.*

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Proof:

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Wave-operators and S-matrix

1. $\mathcal{H}_1 \subset \mathcal{H}$ - single-particle subspace.
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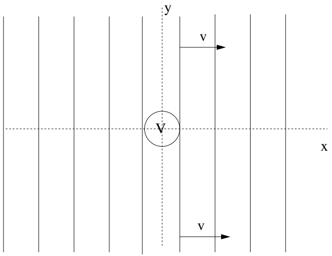

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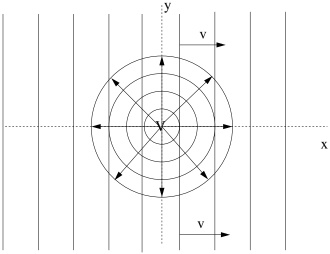

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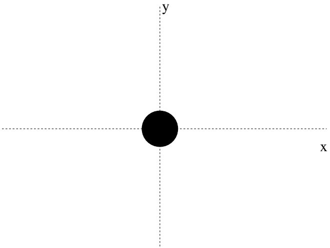
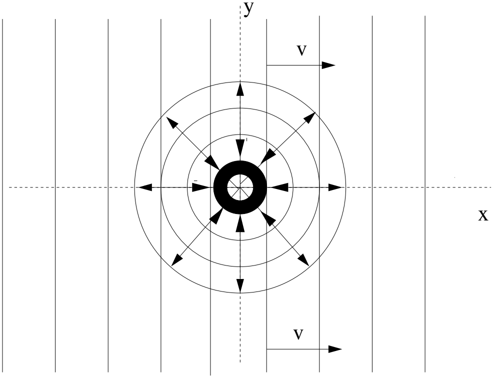


Proving asymptotic completeness in QM

Excluding 'fuzzy' configurations in which the particle cannot decide between a bound state and a scattering state.



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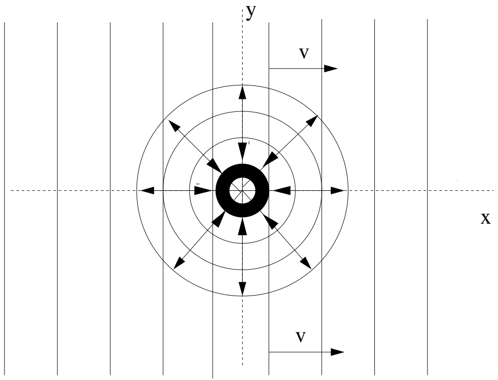

A proof of asymptotic completeness is available in N-body QM [Faddeev 63,..., Sigal-Soffer 87, Graf 90, Dereziński 93]

Asymptotic completeness in QFT and spin systems

1. In systems with infinitely many degrees of freedom there is an additional difficulty:
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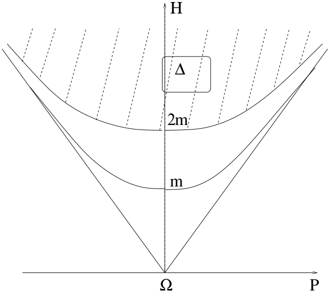

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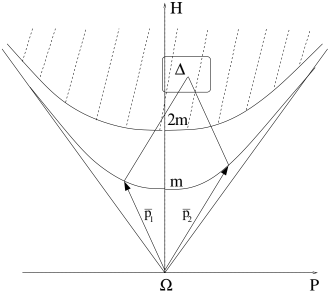

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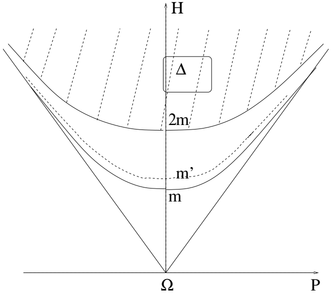

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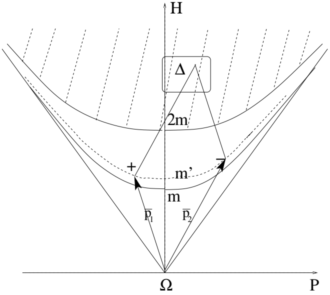
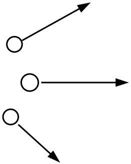


Generalized asymptotic completeness

1. Conventional asymptotic completeness:

$$|\Psi\rangle = |\Psi^{\text{out}}\rangle$$



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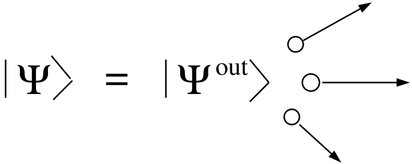
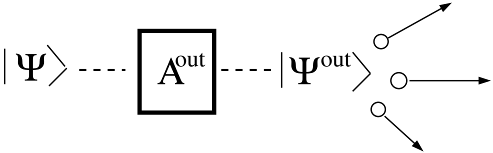


2. Generalized asymptotic completeness [C. Gérard-W.D. 16]:



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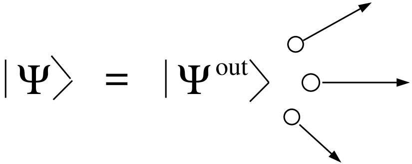

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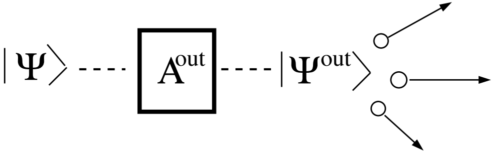

3. Fact. Generalized asymptotic completeness holds under our assumptions.

Araki-Haag detectors

Theorem (Araki-Haag 67, Buchholz 90)

Let $C_t := \int_\Gamma d\mu(x)\tau_{(t,x)}(B^*B)h\big(\frac{x}{t}\big)$, $h \in C_0^\infty(\mathbb{R}^3)$.

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Let $\Delta_1 + \cdots + \Delta_n \subset \Delta$ and h_i have disjoint supports. Then the following limits exist

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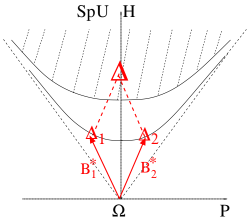
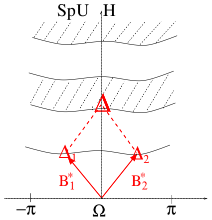
Also, generalized AC holds, i.e. $\mathcal{H}^{\mathrm{out}} = [A^{\mathrm{out}}\mathcal{H}] \oplus \mathbb{C}\Omega$.





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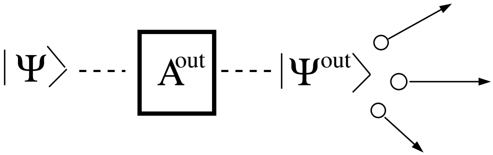

Conclusions and outlook

1. We developed Haag-Ruelle scattering theory for a class of gapped quantum spin systems.
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