Spectral properties of some functional-difference operators for mirror curves

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Weyl operators

We consider a class of functional discrete operators defined as follows:

let

$$U\psi(x) = \psi(x+ib) \qquad \text{and} \qquad V\psi(x) = e^{2\pi b\, x}\psi(x), \quad b>0.$$

Then

$$UV\psi(x) = e^{2\pi b(x+ib)}\psi(x+ib) = e^{2\pi b^2 i}VU\psi(x) = q^2 VU\psi(x), \quad q = e^{\pi b^2 i}.$$

The respective domains of these operators are

$$D(U) = \left\{\psi \in L^2(\mathbb{R}) :\, e^{-2\pi b\,\xi}\,\widehat{\psi}(\xi) \in L^2(\mathbb{R})\right\}$$

and

$$D(V) = \left\{\psi \in L^2(\mathbb{R}) :\, e^{2\pi b\, x}\,\psi(x) \in L^2(\mathbb{R})\right\}.$$

Equivalently, $D(U)$ consists of those functions $\psi(x)$ which admit an analytic continuation to the strip

$$\{z = x + iy \in \mathbb{C} :\ 0 < y < b\}$$

such that $\psi(x + iy) \in L^2(\mathbb{R})$ for all $0 \leq y < b$ and there is a limit

$$\psi(x + ib - i0) = \lim_{\varepsilon \to 0^+} \psi(x + ib - i\varepsilon)$$

in the sense of convergence in $L^2(\mathbb{R})$, which we will denote by $\psi(x + ib)$.

The domains of U^{-1} and V^{-1}can be characterised similarly and obviously

$$U^{-1}\psi(x) = \psi(x - ib) \qquad \text{and} \qquad V^{-1}\psi(x) = e^{-2\pi b\, x}\psi(x).$$

One of the main objects of study is the operator H

$$H = U + U^{-1} + V + V^{-1}$$

whose symbol is $2\cosh(2\pi b\xi) + 2\cosh(2\pi bx)$.

Remark.

It was discovered by M. Aganagic, R. Dijkgraaf, A. Klemm, M. Marino, and C. Vafa, that the functional-difference operators built from the Weyl operators U and V, appear in the study of local mirror symmetry as a quantisation of an algebraic curve, the mirror to a toric Calabi-Yau threefold. The spectral properties of these operators were considered in A. Grassi, Y. Hatsuda, and M. Marino.

Remark.

The operator

$$\mathcal{H}\psi(x) = (U + U^{-1} + V)\psi(x) = \psi(x+ib) + \psi(x-ib) + e^{2\pi b\, x}\psi(x)$$

first appeared in the study of the quantum Liouville model on the lattice and plays an important role in the representation theory of the non-compact quantum group $SL_q(2;\mathbb{R})$. In the momentum representation it becomes the Dehn twist operator in quantum Teichmuller theory.

In particular, R. Kashaev obtained the eigenfunction expansion theorem for this operator in the momentum representation. It was stated as formal completeness and orthogonality relations in the sense of distributions. The spectral analysis of the functional-difference operator $\mathcal{H}$ was studied in the paper of L. D. Faddeev and L. A. Takhtajan. The operator $\mathcal{H}$ was shown to be self-adjoint with a simple absolutely continuous spectrum $[2,\infty)$, and the authors proved eigenfunction expansion theorem for $\mathcal{H}$, by generalizing the classical Kontorovich-Lebedev transform.

Coherent state transform

Let us consider the map $\Phi : L^2(\mathbb{R}) \to L^2(\mathbb{R}^2)$ and defined by

$$\widetilde{\psi}(x,\xi) = (\Phi\,\psi)(x,\xi) = \int_{-\infty}^{\infty} e^{-2\pi i \xi y} g(x-y)\psi(y)\,dy,$$

where

$$g(x) = (1/\pi)^{1/4} e^{-x^2/2}.$$

Note that $\int_{-\infty}^{\infty} g^2(x)dx = 1$ and

$$\begin{aligned}\Phi^*\Phi\,\psi(x) &= \int_{\mathbb{R}^3} e^{2\pi i \xi x} g(x-z) e^{-2\pi i \xi y} g(z-y)\psi(y)\,d\xi dy dz \\ &= \int_{\mathbb{R}^2} \delta(x-y) g(x-z) g(z-y)\,\psi(y)\,dy dz \\ &= \psi(x) \int_{-\infty}^{\infty} g^2(x-z)\,dz = \psi(x).\end{aligned}$$

Theorem.

The map $\Phi : L^2(\mathbb{R}) \to L^2(\mathbb{R}^2)$ is an isometry, such that $\Phi^*\Phi = I$ and $P = \Phi\Phi^*$ is an orthogonal projection in $L^2(\mathbb{R}^2)$.

Coherent state representation for $H = U + U^{-1} + V + V^{-1}$

Theorem. We have a remarkable identity

$$
\begin{aligned}
(H\psi, \psi) &= ((U + U^{-1})\psi, \psi) + ((V + V^{-1})\psi, \psi) \\
&= \iint_{\mathbb{R}^2} 2(d_1 \cosh(2\pi b\, \xi) + d_2 \cosh(2\pi b\, x))|\widetilde{\psi}(x, \xi)|^2 \, d\xi \, dx
\end{aligned}
$$

where

$$d_1 = \frac{2}{((V + V^{-1})\widehat{g}, \widehat{g})} = e^{-b^2/4} < 1,$$

and

$$d_2 = \frac{2}{((V + V^{-1})g, g)} = e^{-(\pi\, b)^2} < 1.$$

Deriving spectral upper bound

Let $\{\lambda_j\}_{j=1}^{\infty}$ be the eigenvalues of H and let $\{\psi_j\}_{j=1}^{\infty}$ be the corresponding orthonormal eigenfunctions which form a complete set. Then using the coherent state representation of H we obtain

$$\sum_{j\geq 1}(\lambda-\lambda_j)_+ = \sum_{j\geq 1}\left(\lambda-(H\psi_j,\psi_j)\right)_+$$

$$= \sum_{j\geq 1}\left(\lambda - \iint_{\mathbb{R}^2} 2\big(d_1\cosh(2\pi b\,\xi)+d_2\cosh(2\pi b\,x)\big)|\widetilde{\psi}_j(x,\xi)|^2\,d\xi\,dy\right)_+ .$$

Note

$$\iint_{\mathbb{R}^2}|\widetilde{\psi}_j(x,\xi)|^2\,dx d\xi = \|\psi_j\|_2^2 = 1.$$

Therefore

$$
\begin{aligned}
&\sum_{j\geq 1}(\lambda-\lambda_j)_+ \\
&\quad = \sum_{j\geq 1}\left(\iint_{\mathbb{R}^2}(\lambda - 2d_1\cosh(2\pi b\,\xi) - 2d_2\cosh(2\pi b\,x))\,|\widetilde{\psi}_j(x,\xi)|^2\,d\xi\,dx\right)_+ \\
&\qquad \leq \iint_{\mathbb{R}^2}\big(\lambda - 2d_1\cosh(2\pi b\,\xi) - 2d_2\cosh(2\pi b\,x)\big)_+\sum_{j\geq 1}|\widetilde{\psi}_j(x,\xi)|^2\,d\xi\,dx\,.
\end{aligned}
$$

Denote $e_{x,\xi}(y) = e^{2\pi i\xi y}g(x-y)$. Since the eigenfunctions ψ_j form an orthonormal basis in $L^2(\mathbb{R})$

$$
\sum_{j=1}^{\infty}|\widetilde{\psi}_j(x,\xi)|^2 = \sum_{j=1}^{\infty}|(e_{x,\xi},\psi_j)|^2 = \|e_{x\xi}\|^2 = 1 \quad \text{for all} \quad x,\xi\in\mathbb{R},
$$

we arrive at the upper bound

$$
\sum_{j\geq 1}(\lambda-\lambda_j)_+ \leq \iint_{\mathbb{R}^2}\big(\lambda - 2d_1\cosh(2\pi b\,\xi) - 2d_2\cosh(2\pi b\,x)\big)_+\,d\xi\,dx\,.
$$

Lower Bound

Using Jensen's inequality for convex functions we also obtain the following lower bound

$$\sum_{j\geq 1}(\lambda-\lambda_j)_+ \geq \iint_{\mathbb{R}^2}\left(\lambda-\frac{2}{d_1}\cosh(2\pi b\,\xi)-\frac{2}{d_2}\cosh(2\pi b\,x)\right)_+ d\xi dx$$
$$= 4\int_0^\infty\int_0^\infty\left(\lambda-\frac{2}{d_1}\cosh(2\pi b\,\xi)-\frac{2}{d_2}\cosh(2\pi b\,x)\right)_+ d\xi dx$$

After simple computations we obtain

Theorem. (L.Schimmer, AL, L.Tahktajanjan, '16)

For the Riesz mean of the eigenvalues of the operator H we have

$$\sum_{j\geq 1}(\lambda-\lambda_j)_+ = \frac{1}{(\pi b)^2}\lambda\log^2\lambda + O(\lambda\log\lambda) \quad \text{as} \quad \lambda\to\infty.$$

Theorem. (L.Schimmer, AL, L.Tahktajanjan, '16)

For the number $N(\lambda)=\#\{j\in\mathbb{N}:\ \lambda_j<\lambda\}$ of eigenvalues of the operator H below λ we have

$$\lim_{\lambda\to\infty}\frac{N(\lambda)}{\log^2\lambda}=\frac{1}{(\pi b)^2}.$$

Remark.

Similar results could be obtained for the operators

$$H(\zeta) = U + U^{-1} + V + \zeta V^{-1}, \qquad \zeta > 0,$$

and

$$H_{m,n} = U + V + q^{-mn} U^{-m} V^{-n}, \quad m, n \in \mathbb{N}.$$

Open Problems.

The spectrum of the problem

$$H\psi(x) = (U + U^{-1} + V + V^{-1})\,\psi(x) \\ = \psi(x + ib) + \psi(x - ib) + 2\cosh(2\pi x b)\,\psi(x) = \lambda\psi(x)$$

is discrete.

- What is the first eigenvalue λ_1.

- As for Harmonic oscillator the operator $U + U^{-1} + V + V^{-1}$ maps by Fourier transform to $V + V^{-1} + U + U^{-1}$.
Is there a suitable factorisation of H as a product of creation and annihilation operators?

- What is the sharp constant C in the inequality

$$\sum_j (\lambda - \lambda_j)_+ \le C\lambda \log^2 \lambda.$$

Free resolvent

Let $b = 1$,

$$H_0\, \psi(x) = (U + U^{-1})\psi(x) = \psi(x+i) + \psi(x-i)$$

and let

$$D\psi(x) = \psi(x+i) - \psi(x) \qquad \text{and} \qquad D^*\psi(x) = \psi(x) - \psi(x-i).$$

Then

$$H_0\, \psi = (D^* D + 2)\psi(x).$$

The generalised eigenfunctions of H_0 are

$$f_\pm(x,k) = e^{\pm 2\pi i k x}.$$

Inddeed, clearly

$$H_0\, f_\pm(x,k) = e^{\pm 2\pi i k(x+i)} + e^{\pm 2\pi i k(x-i)} = 2\cosh(2\pi k)\, f_\pm(x,k). \qquad (*)$$

Therefore it is convenient to use the following parametrisation of the spectral parameter λ as

$$\lambda = 2\cosh(2\pi k).$$

Under this parametrisation the resolvent set $\mathbb{C} \setminus [2, \infty)$ becomes the strip

$$0 < \operatorname{Im} k < 1.$$

Let us introduce the analog of the Wronskian

$$W(f_-, f_+)(x,k) = f_-(x+i,k)\, f_+(x,k) - f_-(x,k)\, f_+(x+i,k)$$
$$= e^{-2\pi ik(x+i)} e^{2\pi ikx} - e^{-2\pi ikx} e^{2\pi ik(x+i)} = 2\sinh(2\pi k),$$

that is also calls the Casorati determinant. Note that

$$\begin{aligned} &D^* W(f_-, f_+)(x,k) \\ &\qquad = f_-(x+i,k)\, f_+(x,k) - f_-(x,k)\, f_+(x+i,k) \\ &\qquad - f_-(x,k)\, f_+(x-i,k) + f_-(x-i,k)\, f_+(x,k) \\ &\qquad\qquad = \lambda(f_-(x,k) f_+(x,k) - f_-(x,k) f_+(x,k)) = 0. \end{aligned}$$

If $\lambda \in \mathbb{C} \setminus [2, \infty)$ then the kernel of the resolvent $R_0 = (H_0 - \lambda)^{-1}$ can be computed by using residue theorem at points $\xi = \pm k$

$$R_0(x-y,\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\xi(x-y)}}{2\cosh(2\pi b\,\xi) - 2\cosh(2\pi b\, k)}\, d\xi$$
$$= \frac{i}{2\sinh(2\pi k)} \left(\frac{e^{2\pi ik(x-y)}}{1 - e^{-2\pi(x-y)}} + \frac{e^{-2\pi ik(x-y)}}{1 - e^{2\pi(x-y)}} \right).$$

This kernel can be also written by using the Jost functions $f_\pm$

We can construct the resolvent $R_0 = (H_0 - \lambda)^{-1}$ by using the Jost function $f_\pm$

$$R_0(x-y,\lambda) = \frac{i}{W(f_-,f_+)(x,k)}\left(\frac{f_-(x,k)f_+(y,k)}{1-e^{2\pi(x-y)}} + \frac{f_-(y,k)f_+(x,k)}{1-e^{-2\pi(x-y)}}\right)$$
$$= \frac{i}{2\sinh(2\pi k)}\left(\frac{e^{-2\pi ik(x-y)}}{1-e^{2\pi(x-y)}} + \frac{e^{-2\pi ik(y-x)}}{1-e^{-2\pi(x-y)}}\right)$$

Consider now

$$R_0(x+i-y-i0,\lambda) + R_0(x-i-y+i0,\lambda) - \lambda\, R_0(x-y,\lambda) = \delta(x-y).$$

Since $f_\pm(x,k)$ satisfy the equation

$$H_0\, f_\pm(x,k) = e^{\pm 2\pi ik(x+i)} + e^{\pm 2\pi ik(x-i)} = 2\cosh(2\pi k)\, f_\pm(x,k)$$

the support of the function on the left hand side is only at $x=y$. Moreover, in the neighbourhood of $x=y$ its singular part coincides with

$$\frac{1}{2\pi i}\left(\frac{1}{x-y-i0} - \frac{1}{x-y+i0}\right) = \delta(x-y).$$

Here we use the fact that the function $\theta(x) = (1-e^{2\pi x})^{-1}$ could be considered as an analogue of the Heaviside function satisfying the identity

$$\theta(x+i-i0) - \theta(x-i+i0) = \delta(x).$$

Remark.

The function $R_0(x,\lambda)$ is regular at $x=0$ and for $0<\operatorname{Im} k<1$ and we have

$$\begin{aligned} |R_0(x,\lambda)| &= \left| \frac{1}{2\sinh(2\pi k)} \left(\frac{e^{-2\pi ikx}}{1-e^{2\pi x}} + \frac{e^{2\pi ikx}}{1-e^{-2\pi x}} \right) \right| \\ &= \left| \frac{1}{\sinh(2\pi k)} \, \frac{\cos(2\pi kx) - \cosh(2\pi x(1+ik))}{1-\cosh(2\pi x)} \right| \\ &\le Ce^{-2\pi \operatorname{Im} k \, |x|}, \qquad x \in \mathbb{R}. \end{aligned}$$

Let us now study the spectral properties of a Schrödinger type operator

$$H = H_0 - V,$$

where V is a complex valued potential. Let $\lambda \notin [2, \infty)$. Such a λ is an eigenvalue of the operator H if and only is the value 1 is an eigenvalue of the Birman-Schwinger operator

$$K = (H_0 - \lambda)^{-1} V.$$

Using the uniform estimate of the resolvent we find

$$1 \leq \|K\|_2^2 = \mathrm{Tr} K^* K \leq C^2 \left(\frac{1}{|\sinh 2\pi k|} \int_{-\infty}^{\infty} |V(x)|\, dx \right)^2 .$$

Note that since $\lambda = 2\cosh(2\pi k)$ and $\sinh(\operatorname{arcosh} t) = \sqrt{t^2 - 1}$ this implies:

Theorem.

Let $\lambda \notin [2, \infty)$ and $V \in L^1(\mathbb{R})$. Then for the complex eigenvalues of the operator H we have the following estimate

$$\left| \sqrt{(\lambda/2)^2 - 1} \right| \leq C \int_{-\infty}^{\infty} |V(x)|\, dx.$$

Question. What is the sharp value of the constant C.

Thank you