Elliptic hypergeometric functions and elliptic difference Painlevé equation

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Abstract

Elliptic hypergeometric functions are a new class of special functions that have been developed during these two decades. In this talk I will give an overview of various aspects of elliptic hypergeometric functions with emphasis on connections with integrable systems including the elliptic difference Painlevé equation.

Plan of this talk

Part 1: Elliptic Hypergeometric Functions

Part 2: Elliptic Difference Painlevé Equation

Part 1: Elliptic Hypergeometric Functions

References for Part 1

[1] M. Ito and M. Noumi: Derivation of a BC_n elliptic summation formula via the fundamental invariants, Constr. Approx. **45** (2017), 33–46 (arXiv:1504.07018, 11 pages).

[2] M. Ito and M. Noumi: Evaluation of the BC_n elliptic Selberg integral via the fundamental invariants, Proc. Amer. Math. Soc. **145** (2017), 689–703 (arXiv:1504.07317, 15 pages).

[3] M. Ito and M. Noumi: A determinant formula associated with the elliptic hypergeometric integrals of type BC_n (in preparation).

1 q-Hypergeometric integrals of Selberg type

○ Selberg integral (1942)

Generalization of the beta integral to a multiple integral involving a power of the difference product (Atle Selberg, 1917–2007):

$$\frac{1}{n!}\int_0^1 \cdots \int_0^1 \prod_{i=1}^{n} z_i^{\alpha-1}(1-z_i)^{\beta-1} \prod_{1\le i<j\le n} |z_i - z_j|^{2\gamma}\, dz_1\cdots dz_n$$

$$= \prod_{j=1}^{n} \frac{\Gamma(\alpha+(j-1)\gamma)\,\Gamma(\beta+(j-1)\gamma)\,\Gamma(j\gamma)}{\Gamma(\alpha+\beta+(n+j-2)\gamma)\,\Gamma(\gamma)}$$

Variations and extensions of this formula, including the cases of integrals of trigonometric and elliptic fuctions, provide with foundations for a variety of theories of hypergeometric functions in many variables.

- Hypergeometric integral of Selberg type = Selberg integral in the broad sense
  Integral of powers of polynomials which involves a power of a difference product or a Weyl denominator
- Selberg integral in the narrow sense
  Hypergeometric integral of Selberg type which admits an evaluation formula in terms of the gamma function

○ q-Hypergeometric integrals of Selberg type

$z = (z_1, \ldots, z_n)$: coordinates of the n-dimensional algebraic torus $\mathbb{T}^n = (\mathbb{C}^*)^n$

There are two types of q-hypergeometric integrals (with base $q \in \mathbb{C}^*$, $|q| < 1$):
Jackson integrals /infinite multiple series (Aomoto–Ito),
versus ordinary integrals over n-cycles in $\mathbb{T}^n$ (Macdonald)

• **Jackson integral:** With a base point $\zeta = (\zeta_1, \ldots, \zeta_n) \in (\mathbb{C}^*)^n$, the Jackson integral of a function $\varphi(z)$ is defined as the infinite multiple series

$$\frac{1}{(1-q)^n} \int_0^{\zeta_1\infty} \cdots \int_0^{\zeta_n\infty} \varphi(z_1, \ldots, z_n) \frac{d_qz_1 \cdots d_qz_n}{z_1 \cdots z_n} = \sum_{\nu_1=-\infty}^{\infty} \cdots \sum_{\nu_n=-\infty}^{\infty} \varphi(q^{\nu_1}\zeta_1, \ldots, q^{\nu_n}\zeta_n).$$

In the notation of multi-indices $\nu = (\nu_1, \ldots, \nu_n) \in \mathbb{Z}^n$, $q^\nu\zeta = (q^{\nu_1}\zeta_1, \ldots, q^{\nu_n}\zeta_n) \in (\mathbb{C}^*)^n$,

$$\int_0^{\zeta\infty} \varphi(z)\,\omega_q(z) = \sum_{\nu\in\mathbb{Z}^n} \varphi(\zeta q^\nu), \quad \omega_q(z) = \frac{1}{(1-q)^n} \frac{d_qz_1 \cdots d_qz_n}{z_1 \cdots z_n}.$$

Sum of the values of $\varphi(z)$ over the multiplicative lattice $\Lambda_\zeta = q^{\mathbb{Z}^n}\zeta \subset (\mathbb{C}^*)^n$.

• **Ordinary integral over an n-cycle:**

$$\int_C \varphi(z)\,\omega(z) = \frac{1}{(2\pi\sqrt{-1})^n} \int_C \varphi(z_1, \ldots, z_n) \frac{dz_1 \cdots dz_n}{z_1 \cdots z_n}, \quad \omega(z) = \frac{1}{(2\pi\sqrt{-1})^n} \frac{dz_1 \cdots dz_n}{z_1 \cdots z_n}$$

Typically, the real torus $\mathbb{T}^n_{\mathbb{R}} = \{|z_1| = \cdots = |z_n| = 1\}$ is chosen for the n-cycle C.

○ q-Shifted factorials

• q-Shifted factorials:

$$(z;q)_\infty = \prod_{i=0}^{\infty}(1-q^i z), \quad (z;q)_k = \frac{(z;q)_\infty}{(q^k z;q)_\infty} \quad (k \in \mathbb{Z})$$

For $k = 0, 1, 2, \ldots$,

$$(z;q)_k = (1-z)(1-qz)\cdots(1-q^{k-1}z), \quad (z;q)_{-k} = \frac{1}{(1-q^{-k}z)(1-q^{-k+1}z)\cdots(1-q^{-1}z)}.$$

q-Shifted factorials are regarded as counterparts of *power functions* or *gamma functions*:

$$\frac{(q^\beta z;q)_\infty}{(q^\alpha z;q)_\infty} \quad \to \quad (1-z)^{\alpha-\beta}; \qquad \frac{(q;q)_\infty}{(q^s;q)_\infty}(1-q)^{1-s} \quad \to \quad \Gamma(s)$$

For $k \in \mathbb{Z}$ or $k = \infty$, a product of q-shifted factorials are often abbreviated as

$$(a_1, \ldots, a_r; q)_k = (a_1;q)_k \cdots (a_r;q)_k.$$

○ q-Beta and q-hypergeometric integrals (contour integrals)

- **Askey–Wilson q-beta integral:** double sign: $f(z^{\pm 1}) = f(z)f(z^{-1})$

$$\frac{1}{2\pi\sqrt{-1}}\int_C \frac{(z^{\pm 2};q)_\infty}{(az^{\pm 1}, bz^{\pm 1}, cz^{\pm 1}, dz^{\pm 1};q)_\infty}\frac{dz}{z} = \frac{2}{(q;q)_\infty}\frac{(abcd;q)_\infty}{(ab,ac,ad,bc,bd,cd;q)_\infty}$$

C: a closed curve separating the poles accumulating at $z=0$ and those at $z=\infty$.

- **Nassrallah–Rahman q-beta integral:** Under the condition $a_0a_1\cdots a_5 = q$,

$$\frac{1}{2\pi\sqrt{-1}}\int_C \frac{(z^{\pm 2};q)_\infty (qa_0^{-1}z^{\pm 1};q)_\infty}{\prod_{k=1}^{5}(a_k z^{\pm 1};q)_\infty}\frac{dz}{z} = \frac{2}{(q;q)_\infty}\frac{\prod_{i=1}^{5}(q/a_ia_0;q)_\infty}{\prod_{1\le i<j\le 5}(a_ia_j;q)_\infty}$$

- **Rahman's q-hypergeometric integral:** (Rahman 1986)

Under the balancing condition $a_0a_1\cdots a_7 = q^2$,

$$\prod_{1\le i<j\le 6}(a_ia_j;q)_\infty \cdot \frac{(q;q)_\infty}{4\pi\sqrt{-1}}\int_C \frac{(z^{\pm 2};q)_\infty \prod_{i=0,7}(qa_i^{-1}z^{\pm 1};q)_\infty}{\prod_{i=1}^{6}(a_iz^{\pm 1};q)_\infty}\frac{dz}{z}$$

$$= \frac{\prod_{i=1}^{6}(qa_i/a_0;q)_\infty (q/a_ia_7;q)_\infty}{(q^2a_0^2;q)_\infty (a_0/a_7;q)_\infty}\,{}_{10}W_9\big(q/a_0^2; q/a_0a_1, q/a_0a_2, \ldots, q/a_0a_7; q, q\big)$$

$$+ \frac{\prod_{i=1}^{6}(qa_i/a_7;q)_\infty (q/a_ia_0;q)_\infty}{(q^2a_7^2;q)_\infty (a_7/a_0;q)_\infty}\,{}_{10}W_9\big(q/a_7^2; q/a_1a_7, q/a_2a_7, \ldots, q/a_6a_7; q, q\big).$$

$${}_{r+3}W_{r+2}\big(a_0; a_1, \ldots, a_r; q, z\big) = \sum_{k=0}^{\infty}\frac{1-q^{2k}a_0}{1-a_0}\frac{(a_0;q)_k}{(q;q)_k}\prod_{i=1}^{r}\frac{(a_i;q)_k}{(qa_0/a_i;q)_k}\ z^k$$

○ q-Hypergeometric integral of Selberg type

$z=(z_1,\ldots,z_n)$: coordinates of the algebraic torus $\mathbb{T}^n=(\mathbb{C}^*)^n$

• **Gustafson's q-Selberg integral** (1990)

[Askey–Wilson] For generic complex parameters $a=(a_1,\ldots,a_4)$ and t,

$$\frac{1}{(2\pi\sqrt{-1})^n}\int_{C^n}\prod_{i=1}^{n}\frac{(z_i^{\pm 2};q)_\infty}{\prod_{k=1}^{4}(a_k z_i^{\pm 1};q)_\infty}\prod_{1\le i<j\le n}\frac{(z_i^{\pm 1}z_j^{\pm 1};q)_\infty}{(tz_i^{\pm 1}z_j^{\pm 1};q)_\infty}\frac{dz_1\cdots dz_n}{z_1\cdots z_n}$$

$$=\frac{2^n n!}{(q;q)_\infty^n}\prod_{i=1}^{n}\left(\frac{(t;q)_\infty}{(t^i;q)_\infty}\frac{(a_1a_2a_3a_4t^{n+i-2};q)_\infty}{\prod_{1\le k<l\le 4}(t^{i-1}a_ka_l;q)_\infty}\right)$$

The integrand is the weight function for the *Koornwinder polynomials* (BC_n).

[Nassrallah-Rahman] Under the balancing condition $a_0a_1a_2a_3a_4a_5t^{2n-2}=q^2$,

$$\frac{1}{(2\pi\sqrt{-1})^n}\int_{C^n}\prod_{i=1}^{n}\frac{(z_i^{\pm 2};q)_\infty(qa_0^{-1}z_i^{\pm 1};q)_\infty}{\prod_{k=1}^{5}(a_k z_i^{\pm 1};q)_\infty}\prod_{1\le i<j\le n}\frac{(z_i^{\pm 1}z_j^{\pm 1};q)_\infty}{(tz_i^{\pm 1}z_j^{\pm 1};q)_\infty}\frac{dz_1\cdots dz_n}{z_1\cdots z_n}$$

$$=\frac{2^n n!}{(q;q)_\infty^n}\prod_{i=1}^{n}\left(\frac{(t;q)_\infty}{(t^i;q)_\infty}\frac{\prod_{k=1}^{5}(t^{1-i}q/a_0a_k;q)_\infty}{\prod_{1\le k<l\le 5}(t^{i-1}a_ka_l;q)_\infty}\right)$$

2 Elliptic hypergeometric integrals of Selberg type

○ Ruijsenaars' elliptic gamma function

With two (generic) bases $p, q \in \mathbb{C}^*$, $|p| < 1, |q| < 1$,

$$\Gamma(z; p, q) = \frac{(pq/z; p, q)_\infty}{(z; p, q)_\infty}, \quad (z; p, q)_\infty = \prod_{i,j=0}^{\infty} (1 - p^i q^j z).$$

It is a meromorphic function on $\mathbb{C}^*$ with simple poles at $z = p^{-i} q^{-j}$ $(i, j = 0, 1, \ldots)$.

- Jacobi theta function (in the multiplicative variable):

$$\theta(z; p) = (z; p)_\infty (p/z; p)_\infty; \quad \theta(pz; p) = -z^{-1}\theta(z; p), \quad \theta(p/z; p) = \theta(z; p)$$

- The elliptic gamma function satisfies the following functional equations:

$$\Gamma(qz; p, q) = \theta(z; p)\Gamma(z; p, q), \quad \Gamma(pq/z; p, q) = \Gamma(z; p, q)^{-1}$$

- In the double sign notation $f(z^{\pm 1}) = f(z)f(z^{-1})$,

$$\begin{aligned}\frac{1}{\Gamma(z^{\pm 1}; p, q)} &= \frac{(z^{\pm 1}; p, q)_\infty}{(pqz^{\pm 1}; p, q)_\infty} = (1 - z^{\pm 1})(pz^{\pm 1}; p)_\infty (qz^{\pm 1}; q)_\infty \\ &= -z^{-1}(z, p/z; p)_\infty (z, q/z; q)_\infty = -z^{-1}\theta(z; p)\theta(z; q)\end{aligned}$$

  holomorphic on $\mathbb{C}^*$, splits into the product of two theta functions with bases p, q.

- In the limit as $p \to 0$,

$$\theta(z; p) \to (1 - z), \quad \Gamma(z; p, q) \to \frac{1}{(z; q)_\infty}, \quad \Gamma(pz; p, q) \to (q/z; q)_\infty$$

○ Elliptic hypergeometric integral of Selberg type (BC_n)

• Elliptic beta integral (Spiridonov 2001)

Under the balancing condition $a_1 \cdots a_6 = pq$,

$$\frac{(p;p)_\infty (q;q)_\infty}{4\pi\sqrt{-1}} \int_C \frac{\prod_{k=1}^{6} \Gamma(a_k z^{\pm 1}; p, q)}{\Gamma(z^{\pm 2}; p, q)} \frac{dz}{z} = \prod_{1 \le k < l \le 6} \Gamma(a_k a_l; p, q)$$

- Elliptic extension of the Nassrallah–Rahman q-beta integral
- Integral version of the Frenkel–Turaev sum

• Elliptic hypergeometric integral of Selberg type

The following integral is called the BC_n *elliptic hypergeometric integral of Selberg type*:

$$I_n(a) = \int_{C^n} \Phi(z;a)\omega(z), \quad \omega(z) = \frac{1}{(2\pi\sqrt{-1})^n} \frac{dz_1 \cdots dz_n}{z_1 \cdots z_n}$$

$$\Phi(z;a) = \prod_{i=1}^{n} \frac{\prod_{k=1}^{m} \Gamma(a_k z_i^{\pm 1}; p, q)}{\Gamma(z_i^{\pm 2}; p, q)} \prod_{1 \le i < j \le n} \frac{\Gamma(t z_i^{\pm 1} z_j^{\pm 1}; p, q)}{\Gamma(z_i^{\pm 1} z_j^{\pm 1}; p, q)}$$

$$a = (a_1, \ldots, a_m) \in (\mathbb{C}^*)^m, \quad t \in \mathbb{C}^*$$

· When $|a_k| < 1$ $(k = 1, \ldots, m)$, $|t| < 1$, a standard choice for the n-cycle C^n is the real torus $\mathbb{T}^n_{\mathbb{R}} = \{|z_1| = \cdots = |z_n| = 1\}$. When the parameters go out from this domain, the n-cycle should be deformed accordingly.

• **Elliptic Selberg integral ($m=6$)** (van Diejen-Spiridonov 2001, Rains)
Under the balancing condition $a_1 \cdots a_6 t^{2n-2} = pq$,

$$I_n(a_1,\ldots,a_6) = \frac{1}{(2\pi\sqrt{-1})^n}\int_{C^n}\prod_{i=1}^{n}\frac{\prod_{k=1}^{6}\Gamma(a_k z_i^{\pm1};p,q)}{\Gamma(z_i^{\pm2};p,q)}\prod_{1\le i<j\le n}\frac{\Gamma(tz_i^{\pm1}z_j^{\pm1};p,q)}{\Gamma(z_i^{\pm1}z_j^{\pm1};p,q)}\frac{dz_1\cdots dz_n}{z_1\cdots z_n}$$

$$= \frac{2^n n!}{(p;p)_\infty^n (q;q)_\infty^n}\prod_{i=1}^{n}\left(\frac{\Gamma(t^i;p,q)}{\Gamma(t;p,q)}\prod_{1\le k<l\le 6}\Gamma(t^{i-1}a_k a_l;p,q)\right)$$

(Elliptic extension of Gustafson's q-Selberg integral)

• **BC_n elliptic hypergeometric integral ($m=8$)** (Rains)

$$I_n(a_1,\ldots,a_8) = \frac{1}{(2\pi\sqrt{-1})^n}\int_{C^n}\prod_{i=1}^{n}\frac{\prod_{k=1}^{8}\Gamma(a_k z_i^{\pm1};p,q)}{\Gamma(z_i^{\pm2};p,q)}\prod_{1\le i<j\le n}\frac{\Gamma(tz_i^{\pm1}z_j^{\pm1};p,q)}{\Gamma(z_i^{\pm1}z_j^{\pm1};p,q)}\frac{dz_1\cdots dz_n}{z_1\cdots z_n}$$

· The Ruijsenaars–van Diejen difference operator of type BC_n is formally selfadjoint with respect to the scalar product defined by the weight function $\Phi(z)$.

· When $t=q$, the sequence of integrals $I_n(a_1,\ldots,a_8)$ ($n=0,1,2,\ldots$) provides with a *hypergeometric τ-function* of the E_8 elliptic difference Painlevé equation (Rains 2005, Noumi 2018). In this case, $I_n(a_1,\ldots,a_8)$ can also be expressed as an $n\times n$ Casorati deteminant whose entries are elliptic hypergeometric integrals in one variable.

3 Determinant of the elliptic hypergeometric integrals

○ General setting of type BC_n

We consider the meromorphic function

$$\Phi(z;a)=\prod_{i=1}^{n}\frac{\prod_{k=1}^{m}\Gamma(a_k z_i^{\pm 1};p,q)}{\Gamma(z_i^{\pm 2};p,q)}\prod_{1\le i<j\le n}\frac{\Gamma(t z_i^{\pm 1} z_j^{\pm 1};p,q)}{\Gamma(z_i^{\pm 1} z_j^{\pm 1};p,q)}$$

of n variables $z=(z_1,\ldots,z_n)\in(\mathbb{C}^*)^n$ with generic parameters $a=(a_1,\ldots,a_m)$ and t. The BC_n elliptic hypergeometric integral (of type II) is defined by

$$I_n(a)=\int_{C^n}\Phi(z;a)\omega(z),\quad \omega(z)=\frac{1}{(2\pi\sqrt{-1})^n}\frac{dz_1\cdots dz_n}{z_1\cdots z_n}.$$

The integrand $\Phi(z;a)$ is invariant under the action of the Weyl group $W_n=\{\pm 1\}^n\rtimes\mathfrak{S}_n$ of type BC_n (hyperoctahedral group of degree n).

○ Bilinear form defined by the integral

Assuming that $m = 2r+4$ (even), we denote by $\mathcal{H}^{(p)}_{r-1}$ the $\mathbb{C}$-vector space of W_n-invariant holomorphic functions of degree $r-1$ with respect to p:

$$\mathcal{H}^{(p)}_{r-1,n} = \left\{ f \in \mathcal{O}((\mathbb{C}^*)^n)^{W_n} \;\middle|\; T_{p,z_i} f(z) = f(z)(pz_i^2)^{-r+1} \quad (i = 1, \ldots, n) \right\}.$$

$$\dim_{\mathbb{C}} \mathcal{H}^{(p)}_{r-1,n} = \binom{n+r-1}{r-1}.$$

Taking the two $\mathbb{C}$-vector spaces $\mathcal{H}^{(p)}_{r-1,n}$, $\mathcal{H}^{(q)}_{r-1,n}$ for the two bases p, q, respectively, we introduce the *hypergeometric pairing* (following the terminology of Tarasov-Varchenko)

$$\langle \cdot, \cdot \rangle_\Phi : \; \mathcal{H}^{(p)}_{r-1,n} \times \mathcal{H}^{(q)}_{r-1,n} \to \mathbb{C},$$

$$\langle \varphi(z), \psi(z) \rangle_\Phi = \int_{C^n} \varphi(z)\psi(z)\Phi(z)\omega(z) \quad (\varphi \in \mathcal{H}^{(p)}_{r-1,n}, \; \psi \in \mathcal{H}^{(q)}_{r-1,n})$$

associated with the integral with respect to $\Phi(z) = \Phi(z; a)$.

In this setting the vector space $\mathcal{H}^{(p)}_{r-1,n}$ can be regarded as the space of *n-cocycles* representing the W_n-invariant q-difference de Rham cohomology associated with $\Phi(z)$. The vector space $\mathcal{H}^{(q)}_{r-1,n}$ in turn plays the role of the space of *n-cycles* for this q-difference de Rham cohomology. Note that the dimension

$$\dim_{\mathbb{C}} \mathcal{H}^{(p)}_{r-1,n} = \dim_{\mathbb{C}} \mathcal{H}^{(q)}_{r-1,n} = \binom{n+r-1}{r-1}$$

is 1 for $r = 1$, and $n+1$ for $r = 2$.

Note that the dimension $\binom{n+r-1}{r-1}$ of $\mathcal{H}^{(p)}_{r-1,n}$ coincides with the cardinality of the set of multiindices

$$Z_{r,n} = \left\{ \mu = (\mu_1, \ldots, \mu_r) \in \mathbb{N}^r \;\middle|\; |\mu| = \mu_1 + \cdots + \mu_r = n \right\}.$$

Choosing generic r parameters $x = (x_1, \ldots, x_r) \in (\mathbb{C}^*)^r$, we consider the set of reference points $(x)_{t,\nu}$ $(\nu \in Z_{r,n})$ in $(\mathbb{C}^*)^n$ defined by *multiple principal specialization*:

$$(x)_{t,\nu} = (x_1, tx_1, \ldots, t^{\nu_1 - 1}x_1;\ x_2, tx_2, \ldots, t^{\nu_2 - 1}x_2;\ \ldots) \in (\mathbb{C}^*)^n \quad (r \text{ blocks}).$$

Then one can show that $\mathcal{H}^{(p)}_{r-1,n}$ has a unique *interpolation function basis* such that

$$E_\mu(x; (x)_{t,\nu}; p) = \delta_{\mu,\nu} \quad (\mu, \nu \in Z_{r,n}).$$

Using the two kinds of interpolation functions with bases p, q respectively, we define the integrals

$$\begin{aligned} K_{\mu,\nu}(a; x, y) = K_{\mu,\nu}(a; x, y; p, q) &= \langle E_\mu(x; z; p), E_\nu(y; z; q) \rangle_\Phi \\ &= \int_{C^n} E_\mu(x; z; p) E_\nu(y; z; q) \Phi(a; z; p, q) \omega(z) \quad (\mu, \nu \in Z_{r,n}). \end{aligned}$$

The $\binom{n+r-1}{r-1} \times \binom{n+r-1}{r-1}$ matrix $K^{(r,n)}(a; x, y) = \big(K_{\mu,\nu}(a; x, y)\big)_{\mu,\nu \in Z_{r,n}}$ is the representation matrix of the hypergeometric pairing

$$\langle \cdot, \cdot \rangle_\Phi : \mathcal{H}^{(p)}_{r-1,n} \times \mathcal{H}^{(q)}_{r-1,n} \to \mathbb{C}; \quad \langle \varphi(z), \psi(z) \rangle_\Phi = \int_{C^n} \varphi(z) \psi(z) \Phi(z) \omega(z)$$

in terms of the interpolation bases.

We assume below that the balancing condition $a_1\cdots a_m t^{2n-2}=pq$ is satisfied.

Theorem A: *The matrix $K^{(r,n)}(a;x,y)$ satisfies a system of first order q-difference and p-difference equations of the form*

$$T_{q,a_k}T_{q,a_l}^{-1}K^{(r,n)}(a;x,y)=A_{k,l}(a;x,y)\,K^{(r,n)}(a;x,y)\quad(1\le k<l\le m),$$
$$T_{p,a_k}T_{p,a_l}^{-1}K^{(r,n)}(a;x,y)=K^{(r,n)}(a;x,y)\,B_{k,l}(a;x,y)\quad(1\le k<l\le m).$$

We remark that $B_{k,l}(a;x,y)$ is obtained as the transposed matrix of $A_{k,l}(a;x,y)$ with the roles of (x,y) and (p,q) exchanged.

The matrix $K^{(r,n)}(a;x,y)$ can be thought of as a fundamental system of solutions of the q-difference/p-difference systems. Also, non-degeneracy of the hypergeometric pairing is guaranteed by an explicit evaluation formula for the determinant of $K^{(r,n)}(a;x,y)$.

Theorem B: *The determinant of the matrix $K^{(r,n)}(a;x,y)$ is evaluated as follows:*

$$\det K^{(r,n)}(a;x,y)=c^{(r,n)}L^{(r,n)}(a;x,y)$$

$$L^{(r,n)}(a;x,y)=\frac{\prod_{i=0}^{n-1}\prod_{1\le k<l\le m}\Gamma(t^ia_ka_l;p,q)^{\binom{n-i+r-2}{r-1}}}{\prod_{0\le i+j<n}\prod_{1\le k<l\le r}\left(e(t^ix_k,t^jx_l;p)e(t^iy_k,t^jy_l;q)\right)^{\binom{n-i-j+r-3}{r-2}}}$$

$$c^{(r,n)}=\left(\frac{2^nn!}{(p;p)_\infty^n(q;q)_\infty^n}\right)^{\binom{n+r-1}{r-1}}\frac{\prod_{i=1}^{n}\Gamma(t^i;p,q)^{r\binom{n-i+r-1}{r-1}}}{\Gamma(t;p,q)^{r\binom{n+r-1}{r}}},$$

where $e(u,v;p)=u^{-1}\theta(uv;p)\theta(u/v;p)$.

- $\boldsymbol{r = 1}$ **(**$\boldsymbol{m = 6}$**):** 1×1 determinant (van Diejen–Spiridonov 2001)

$$\det K^{(1,n)}(a) = \frac{2^n n!}{(p;p)_\infty^n (q;q)_\infty^n} \frac{\prod_{i=1}^n \Gamma(t^i;p,q)}{\Gamma(t;p,q)^n} \prod_{i=0}^{n-1} \prod_{1\le k<l\le 6} \Gamma(t^i a_k a_l;p,q)$$

- $\boldsymbol{r = 2}$ **(**$\boldsymbol{m = 8}$**):** $(n+1) \times (n+1)$ determinant

$$\det K^{(2,n)}(a;x,y) = \left(\frac{2^n n!}{(p;p)_\infty^n (q;q)_\infty^n}\right)^{n+1} \frac{\prod_{i=1}^n \Gamma(t^i;p,q)^{2(n-i+1)}}{\Gamma(t;p,q)^{n(n+1)}} \cdot \frac{\prod_{i=0}^{n-1} \prod_{1\le k<l\le 8} \Gamma(t^i a_k a_l;p,q)^{n-i}}{\prod_{0\le i+j<n} e(t^i x_1, t^j x_2;p) e(t^i y_1, t^j y_2;q)}.$$

References for Part 1

[1] K. Aomoto and M. Ito: A determinant formula for a holonomic q-difference system associated with Jackson integrals of type BC_n, Adv. Math. **221**(2009), 1069–1114.

[2] M. Ito and P.J. Forrester, A bilateral extension of q-Selberg integral, Trans. Amer. Math. Soc. **369** (2017), 2843–2878. arXiv:1309.0001, 36 pages.

[3] M. Ito and M. Noumi: Derivation of a BC_n elliptic summation formula via the fundamental invariants, Constr. Approx. **45** (2017), 33-46. (arXiv:1504.07018, 11 pages).

[4] M. Ito and M. Noumi: Evaluation of the BC_n elliptic Selberg integral via the fundamental invariants, Proc. Amer. Math. Soc. **145** (2017), 689–703. (arXiv:1504.07317, 15 pages).

[5] M. Ito and M. Noumi: A generalization of the Sears–Slater transformation and elliptic Lagrange interpolation of type BC_n, Adv. in Math. **229** (2016), 361–380 (arXiv:1506.07267, 17 pages).

[6] M. Ito and M. Noumi: Connection formula for the Jackson integral of type A_n and elliptic Lagrange interpolation, to appear in SIGMA (arXiv:1801.07041, 43 pages)

[7] M. Ito and M. Noumi: A determinant formula associated with the elliptic hypergeometric integrals of type BC_n (in preparation)

Part 2: Elliptic Difference Painlevé Equation

References for Part 2

[1] K. Kajiwara, T. Masuda, M. Noumi, Y. Ohta and Y. Yamada: ${}_{10}E_9$ solution to the elliptic Painlevé equation, J. Phys. A. 36(2003), L263–L272.

[2] K. Kajiwara, T. Masuda, M. Noumi, Y. Ohta and Y. Yamada: Point configurations, Cremona transformations and the elliptic difference Painlevé equation, *Théories asymptotiques et équations de Painlevé* (Angers, juin 2004), Séminaires et Congrès 14(2006), 169–198.

[3] K. Kajiwara, M. Noumi and Y. Yamada: Geometric aspects of Painlevé equations, J. Phys. A: Math. Theor. 50 (2017), 073001 (164pp) (arXiv:1509.08168, 167 pages)

[4] M. Noumi: Remarks on τ-functions for the difference Painlevé equations of type E_8, Advanced Studies in Pure Mathematics **76** (2018), 1–65 (arXiv:16040.6869, 55 pages)

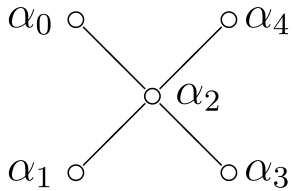
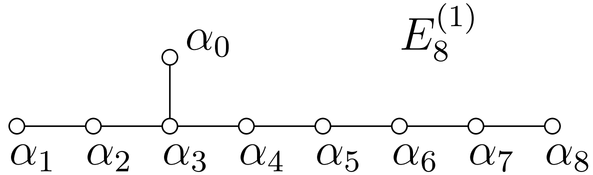
1 Elliptic difference Painlevé equation

○ Elliptic (difference) Painlevé equation

A system of nonlinear difference equations with elliptic function coefficients
with affine Weyl group symmetry of type E_8
... Master equation for "all" second order discrete Painlevé equations

- **Several approaches to the elliptic Painlevé equation**
  - Ohta-Ramani-Grammaticos (J.Phys. A 2001)
    Bilinear equations for τ-functions on the E_8 lattice
  - Sakai (CMP 2001)
    Discrete dynamical system on the rational surface obtained from $\mathbb{P}^2$ by blowingup at nine points (or from $\mathbb{P}^1 \times \mathbb{P}^1$ by blowingup at eight points) in general poisition
  - Kajiwara et al. (J.Phys. A 2003, Séminaires et Congrès 2006)
    τ-Functions associated with Weyl group actions on point configuration spaces
  - Rains (SIGMA 2011), Yamada (IMRN 2011)
    Compatibility condition of linear difference equations (Lax pairs)

• Differential equations *versus* difference equations

	nonlinear	linear
Differential equation	Painlevé VI equation (1 + 4 parameters) $t\ +$  $D_4^{(1)}$	Gauss hypergeometric equation (1 + 3 parameters) ${}_2F_1\left(\begin{matrix} a,\ b \\ c \end{matrix};x\right)$
Difference equation	Elliptic Painlevé equation (8 parameters)  $E_8^{(1)}$	Elliptic hypergeometric equation (7 parameters) ${}_{12}V_{11}\,(a_0; a_1, \ldots, a_7; p, q)$ or $\ I(t_0, t_1, \ldots, t_7; p, q)$

○ Sakai's table of second order discrete Painlevé equations

Sakai's table (2001): a standard list of second order discrete Painlevé equations
classification of nine-point blowups of $\mathbb{P}^2$, or eight-point blowups of $\mathbb{P}^1 \times \mathbb{P}^1$, which admit affine Weyl group symmetries.

– Rational surfaces (anti-canonical divisors)

$$(eP): \quad \boxed{A_0^{(1)}}$$

$$(qP): \quad \begin{array}{l} A_0^{(1)} \to A_1^{(1)} \to A_2^{(1)} \to A_3^{(1)} \to A_4^{(1)} \to A_5^{(1)} \to A_6^{(1)} \to A_7^{(1)} \to A_8^{(1)} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad A_6^{(1)} \searrow A_7'^{(1)} \end{array}$$

$$(dP): \quad \begin{array}{ccccccccccccccc} A_0^{(1)} & \to & A_1^{(1)} & \to & A_2^{(1)} & \to & D_4^{(1)} & \to & D_5^{(1)} & \to & D_6^{(1)} & \to & D_7^{(1)} & \to & D_8^{(1)} \\ & & & & & & & & & \searrow & & \searrow & & & \\ & & & & & & & & & & E_6^{(1)} & \to & E_7^{(1)} & \to & E_8^{(1)} \end{array}$$

– Affine Weyl group symmetry

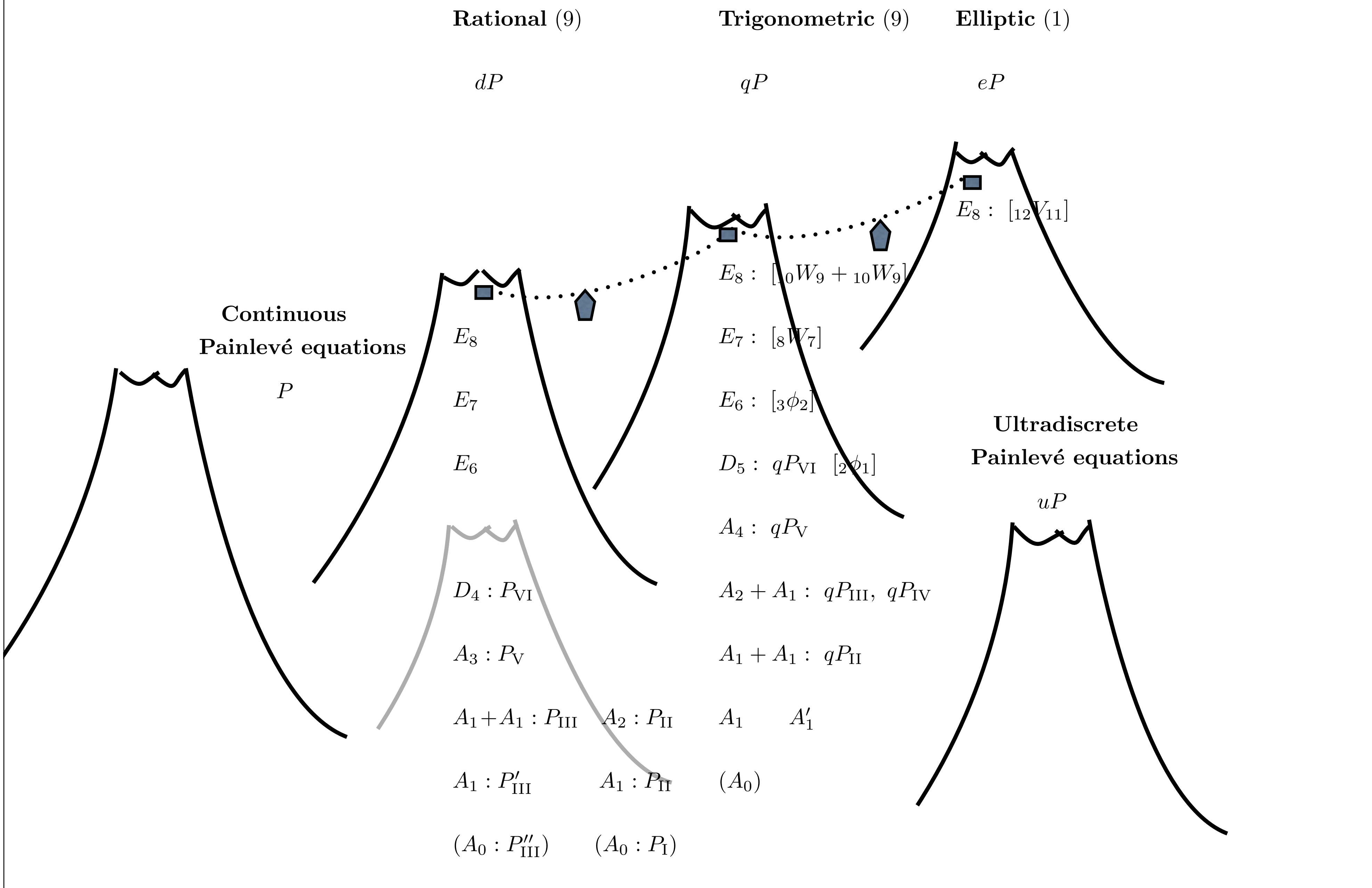
$$(eP): \quad \boxed{E_8^{(1)}}$$

$$(qP): \quad \begin{array}{l} E_8^{(1)} \to E_7^{(1)} \to E_6^{(1)} \to D_5^{(1)} \to A_4^{(1)} \to (A_2 + A_1)^{(1)} \to (A_1 + A_1')^{(1)} \to A_1'^{(1)} \to A_0^{(1)} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad (A_1 + A_1')^{(1)} \searrow A_1^{(1)} \end{array}$$

$$(dP): \quad \begin{array}{ccccccccccccc} E_8^{(1)} & \to & E_7^{(1)} & \to & E_6^{(1)} & \to & D_4^{(1)} & \to & A_3^{(1)} & \to & (2A_1)^{(1)} & \to & A_1'^{(1)} \to A_0^{(1)} \\ & & & & & & & & & \searrow & & \searrow & \\ & & & & & & & & & & A_2^{(1)} & \to & A_1^{(1)} \to A_0^{(1)} \end{array}$$

Discrete Painlevé equations

(Grammaticos-Ramani-$\cdots$ & Sakai)



○ Hermite's theorem

Suppose that a nonzero entire function $s(z)$ ($z \in \mathbb{C}$) satisfies the functional equation of three terms

$$\begin{aligned} s(z+a)s(z-a)s(b+c)s(b-c) + s(z+b)s(z-b)s(c+a)s(c-a) \\ + s(z+c)s(z-c)s(a+b)s(a-b) = 0 \end{aligned}$$

for any $z, a, b, c \in \mathbb{C}$. Then, it is known that

(0) $s(z)$ is an odd function ($s(0) = 0$, $s(-z) = -s(z)$), and

(1) the set of zeros $\Omega = \left\{ a \in \mathbb{C} \;\middle|\; s(a) = 0 \right\}$ is a closed discrete subgroup of $(\mathbb{C}, +)$.

Furthermore, up to multiplication by $\exp(az^2 + c)$ for some $a, c \in \mathbb{C}$, it belongs to one of the following three classes of functions:

$$\begin{array}{llcll} (0) & \textit{rational} & : & s(z) = z & \Omega = 0 \\ (1) & \textit{trigonometric} & : & s(z) = \sin(\pi z/\omega_1) & \Omega = \mathbb{Z}\,\omega_1 \\ (2) & \textit{elliptic} & : & s(z) = \sigma(z|\Omega) & \Omega = \mathbb{Z}\,\omega_1 \oplus \mathbb{Z}\,\omega_2, \end{array}$$

where $\sigma(z|\Omega)$ denotes the Weierstrass sigma function

$$\sigma(z|\Omega) = z \prod_{\omega \in \Omega,\, \omega \neq 0} \left(1 - \frac{z}{\omega}\right) e^{z^2/2\omega + z/\omega}.$$

• Remarks on the three-term relation

[1] A considerable part of the theory of difference equations of Painlevé type and of hypergeometric type can be formulated in terms of an arbitrarily chosen *fundamental function* $s(z)$ satisfying the three-term relation mentioned above, without discriminating the three classes (rational, trigonometric, elliptic). In such a case, it is convenient to use the *e-number notation* $[z] = s(z)$ assuming that

$$[z \pm a][b \pm c] + [z \pm b][c \pm a] + [z \pm c][a \pm c] = 0$$

with abbreviation $[a \pm b] = [a+b][a-b]$. It should be noted that this three-term relation is already a *Hirota equation* in dimension one.

[2] Hereafter, we consider the elliptic case with the additive e-number notation

$$[z] = \sigma(z|\Omega), \quad \Omega = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2.$$

When $\Omega = \mathbb{Z}1 \oplus \mathbb{Z}\tau$ and $\mathrm{Im}\,\tau > 0$, setting $p = e(\tau) = e^{2\pi\sqrt{-1}\tau}$ we define

$$\theta(u;p) = (u;p)_\infty (p/u;p)_\infty, \quad (u;p)_\infty = \prod_{i=0}^{\infty}(1 - p^i u) \quad (|p| < 1).$$

In this *multiplicative* notation of elliptic theta function, the odd Jacobi theta function is expressed as $[z] = \text{const.}\, u^{-\frac{1}{2}}\theta(u;p)$ $(u = e(z))$; this function satisfies the above three-term relation.

○ Elliptic difference Painlevé equation

The elliptic difference Painlevé equation is a system of equations for two dependent variables $(x,y)\in\mathbb{P}^1\times\mathbb{P}^1$ with respect to translations T_α by E_8 roots:

$$(*)\qquad T_\alpha(x)=R_\alpha(x,y),\quad T_\alpha(y)=S_\alpha(x,y)\qquad(\alpha\in Q(E_8))$$

where $R_\alpha(x,y),S_\alpha(x,y)\in\mathcal{K}(x,y)$ are rational functions in (x,y) with coefficients in the field $\mathcal{K}=\mathcal{M}(A)$ of meromorphic functions on the complex torus $A=L\otimes_{\mathbb{Z}}E_\Omega$, with $E_\Omega=\mathbb{C}/\Omega$. Here L denotes the *Picard lattice* associated with the eight-point blowup of $\mathbb{P}^1\times\mathbb{P}^1$:

$$L=\mathbb{Z}\mathsf{h}_1\oplus\mathbb{Z}\mathsf{h}_2\oplus\mathbb{Z}\mathsf{e}_1\oplus\mathbb{Z}\mathsf{e}_2\oplus\cdots\oplus\mathbb{Z}\mathsf{e}_8$$

$$(\mathsf{h}_1|\mathsf{h}_1)=(\mathsf{h}_2|\mathsf{h}_2)=0,\quad(\mathsf{h}_1|\mathsf{h}_2)=-1,\quad(\mathsf{h}_i|\mathsf{e}_j)=0,\quad(\mathsf{e}_i|\mathsf{e}_j)=\delta_{ij}$$

Note that the complexification $\mathfrak{h}=L\otimes_{\mathbb{Z}}\mathbb{C}$ is identified with the Cartan subalgebra of the affine Lie algebra $\mathfrak{g}(E_8^{(1)})$, and that the complex torus is expressed as $A=\mathfrak{h}/L\otimes_{\mathbb{Z}}\Omega$. The elliptic difference equation $(*)$ is then defined through a certain representation of the affine Weyl group $W(E_8^{(1)})=T_Q\rtimes W(E_8)$, $Q=Q(E_8)$ on $\mathcal{K}(x,y)$.

For the elliptic difference Painlevé equation defined as above, one can introduce a family of τ variables $\big(\tau(\Lambda)\big)_{\Lambda\in M}$ indexed by the $W(E_8^{(1)})$-orbit $M = W(E_8^{(1})\mathsf{e}_8 \subset L$ such that

$$\tau(\Lambda) = T_{\mathsf{e}_8-\Lambda}(\tau(\mathsf{e}_8)) \quad (\Lambda \in M)$$

$$x = \frac{\tau(\mathsf{e}_2)\tau(\mathsf{h}_1-\mathsf{e}_2)}{\tau(\mathsf{e}_1)\tau(\mathsf{h}_1-\mathsf{e}_1)}, \quad y = \frac{\tau(\mathsf{e}_2)\tau(\mathsf{h}_2-\mathsf{e}_2)}{\tau(\mathsf{e}_1)\tau(\mathsf{h}_2-\mathsf{e}_1)}.$$

Then the elliptic difference Painlevé equation is translated into a system of non-autonomous Hirota equations of the form

$$\begin{aligned}
&\sigma(\mathsf{e}_j-\mathsf{e}_k)\sigma(\mathsf{h}_r-\mathsf{e}_j-\mathsf{e}_k)\tau(\mathsf{e}_i)\tau(\mathsf{h}_r-E_i)\\
&+\sigma(\mathsf{e}_k-\mathsf{e}_i)\sigma(\mathsf{h}_r-\mathsf{e}_k-\mathsf{e}_i)\tau(\mathsf{e}_j)\tau(\mathsf{h}_r-E_j)\\
&+\sigma(\mathsf{e}_i-\mathsf{e}_j)\sigma(\mathsf{h}_r-\mathsf{e}_i-\mathsf{e}_j)\tau(\mathsf{e}_k)\tau(\mathsf{h}_r-E_k)=0
\end{aligned}$$

involving elliptic theta functions as coefficients, together with their $W(E_8)$- translates. This system of bilinear equations is essentially the same as the τ-functions on the E_8 lattice proposed earliear by Ohta-Ramani-Grammaticos (2001).

Reformmulating this system of non-autonomous Hirota equations, we introduce below the notion of *ORG τ-functions*, and construct *hypergeometric ORG τ-functions* in that framework.

2 $eP(E_8^{(1)})$ as a system of non-autonomous Hirota equations

○ A realization of the root lattice $P = Q(E_8)$

$$V = \mathbb{C}^8 = \mathbb{C}v_0 \oplus \mathbb{C}v_1 \oplus \cdots \oplus \mathbb{C}v_7; \quad (v_i|v_j) = \delta_{ij} \quad (i, j \in \{0, 1, \ldots, 7\}).$$

$$P = \left\{ a \in \mathbb{Z}^8 \cup (\phi + \mathbb{Z}^8) \mid (\phi|a) \in \mathbb{Z} \right\}$$

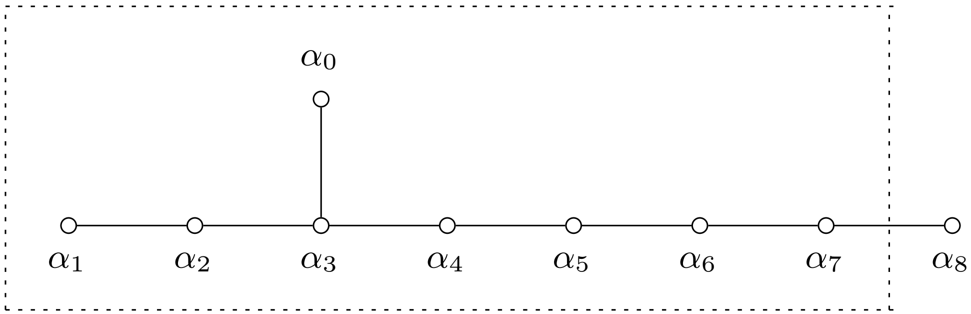
$$\phi = \tfrac{1}{2}(1,1,1,1,1,1,1,1) = \tfrac{1}{2}(v_0 + v_1 + \cdots + v_7)$$

$$\Delta(E_8) = \{ \alpha \in P \mid (\alpha|\alpha) = 2 \}, \quad |\Delta(E_8)| = 240.$$

$(1):\ \pm v_i \pm v_j \quad (0 \le i < j \le 7) \qquad \cdots \ \binom{8}{2} \cdot 4 = 112$

$(2):\ \frac{1}{2}(\pm v_0 \pm \cdots \pm v_7) \quad$ (even number of $-$ signs) $\quad \cdots \ 2^7 = 128$

$$\sum_{a \in P} q^{(a|a)} = 1 + 240q^2 + 2160q^4 + 6720q^6 + 17520q^8 + \cdots$$



$\alpha_0 = \phi - v_0 - v_1 - v_2 - v_3,$

$\alpha_j = v_j - v_{j+1} \quad (j = 1, \ldots, 6)$

$\alpha_7 = v_7 + v_0$

$\alpha_8 = \delta - \phi$

ϕ : highest root of $\Delta(E_8)$

○ C_l-frames in $P = Q(E_8)$

Definition (C_l-frame): For each $l = 1, 2, 3, \ldots$, a set of $2l$ vectors $\{\pm a_1, \ldots, \pm a_l\}$ in V is called a C_l-*frame* if

$$(1) \quad (a_i|a_j) = \delta_{ij} \quad (i, j \in \{1, \ldots, l\}),$$

$$(2) \quad \{ \pm a_i \pm a_j \mid 1 \leq i < j \leq l \} \cup \{ \pm 2a_i \mid 1 \leq i \leq l \} \subset P.$$

There are 2160 vectors $a \in \frac{1}{2}P$ with $(a|a) = 1$. Let $\mathcal{C}_l$ be the set of all C_l frames in P:

$$(\tfrac{1}{2}P)_1 = \bigsqcup_{A \in \mathcal{C}_8} A; \quad |\mathcal{C}_8| = 135, \quad |\mathcal{C}_3| = 135 \cdot \binom{8}{3} = 7560$$

Hereafter we use the notation $[\zeta] = \sigma(\zeta|\Omega)$ or $[\zeta] = z^{-\frac{1}{2}}\theta(z;p)$, $z = e^{2\pi\sqrt{-1}\zeta}$ so that

$$[\beta \pm \gamma][\zeta \pm \alpha] + [\gamma \pm \alpha][\zeta \pm \beta] + [\alpha \pm \beta][\zeta \pm \gamma] = 0.$$

○ ORG $\boldsymbol{\tau}$-function

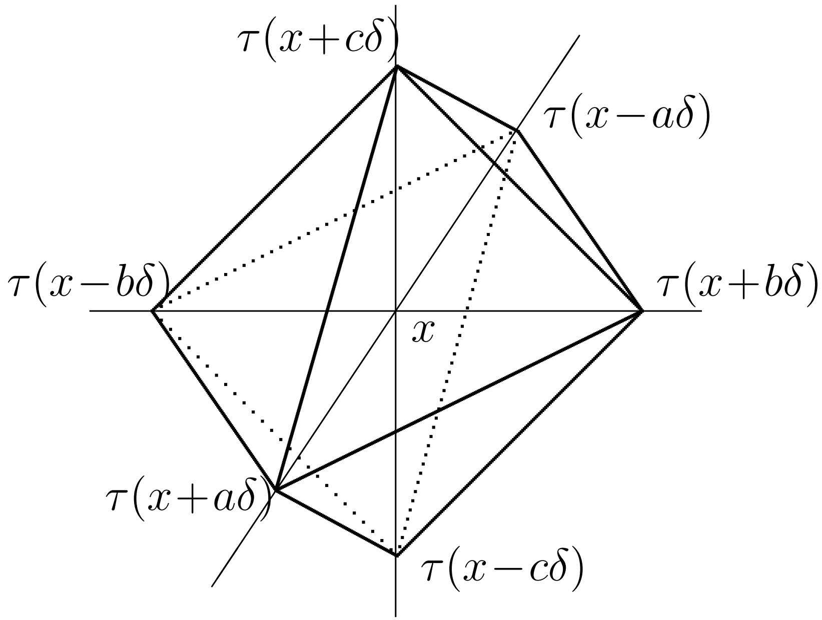
Fix a nonzero constant δ. Let D be a subset of $V = \mathbb{C}^8$ such that $D + P\delta = D$.

Definition (ORG $\boldsymbol{\tau}$-function): A function $\tau(x)$ defined over D is called an *ORG τ-function* if it satisfies the non-autonomous Hirota equation

$$
\begin{aligned}
&[(b+c|x)]\,[(b-c|x)]\,\tau(x+a\delta)\,\tau(x-a\delta)\\
&+[(c+a|x)]\,[(c-a|x)]\,\tau(x+b\delta)\,\tau(x-b\delta)\\
&+[(a+b|x)]\,[(a-b|x)]\,\tau(x+c\delta)\,\tau(x-c\delta)=0
\end{aligned}
$$

for any C_3-frame $\{\pm a, \pm b, \pm c\}$ in $P = Q(E_8)$.

Each of the six points $x \pm a\delta$, $x \pm b\delta$, $x \pm c\delta$ belongs to D if and only if the others do. In this formulation $eP(E_8)$ is a $W(E_8)$-invariant system of 7560 non-autonomous Hirota equations.



○ $eP(E_8)$ τ-function as an infinite chain of $eP(E_7)$ τ-functions

In the E_8 root lattice $P = Q(E_8)$, the E_7 root lattice is realized as

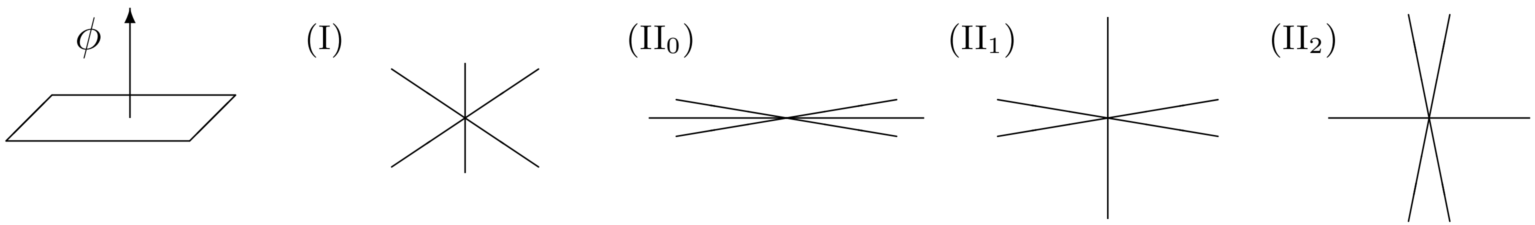
$$Q(E_7) = \{a \in P \mid (\phi|a) = 0\ \} \subset P = Q(E_8); \quad \Delta(E_7) = \Delta(E_8)^{\perp\phi}.$$

Fixing a constant $c \in \mathbb{C}$, we consider the union of parallel hyperplanes

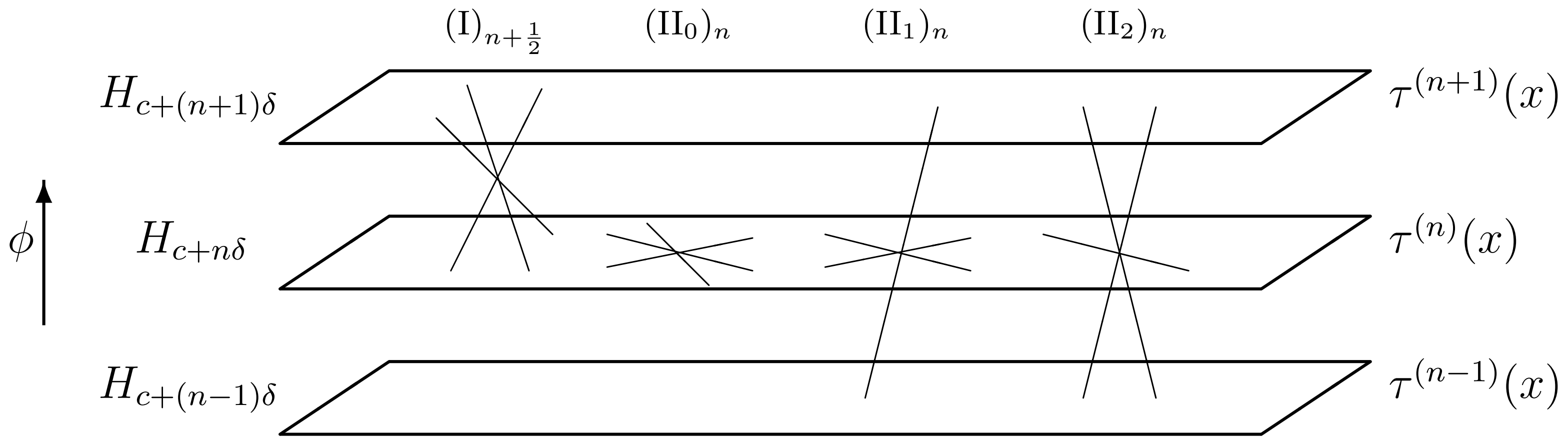
$$D_c = \bigsqcup_{n\in\mathbb{Z}} H_{c+n\delta}; \quad H_{c+n\delta} = \{x \in V \mid (\phi|x) = c + n\delta\} \quad (n \in \mathbb{Z}).$$

Then an ORG τ-function $\tau(x)$ on D_c can be regarded as a chain $\{\tau^{(n)}(x)\}_{n\in\mathbb{Z}}$ of $eP(E_7)$ τ-functions on parallel hyperplanes by setting $\tau^{(n)} = \tau|_{H_{c+n\delta}}$ $(n \in \mathbb{Z})$.

- **Four types of 7560 C_3-frames** (specified by the scalar products with ϕ)



(I)	(II$_0$)	(II$_1$)	(II$_2$)
$56 \cdot 72$	$20 \cdot 63$	$30 \cdot 63$	$6 \cdot 63$
$(\pm\frac{1}{2})^3$	0^6	$(\pm 1)\, 0^4$	$(\pm 1)^2\, 0^2$



• Four types of Hirota bilinear equations

Four types of bilinear equations corresponding to the types I, II_0, II_1, II_2 of C_3-frames:

$$(\mathrm{I})_{n+\frac{1}{2}}:\ [(a_1 \pm a_2|x)]\tau^{(n)}(x-a_0\delta)\tau^{(n+1)}(x+a_0\delta)+\cdots=0$$

$$(\mathrm{II}_0)_n:\ [(a_1 \pm a_2|x)]\tau^{(n)}(x-a_0\delta)\tau^{(n)}(x+a_0\delta)+\cdots=0$$

$$(\mathrm{II}_1)_n:\ [(a_1 \pm a_2|x)]\tau^{(n-1)}(x-a_0\delta)\tau^{(n+1)}(x+a_0\delta)$$
$$=[(a_0 \pm a_2|x)]\tau^{(n)}(x\pm a_1\delta)-[(a_0 \pm a_1|x)]\tau^{(n)}(x\pm a_2\delta)$$

$$(\mathrm{II}_2)_n:\ [(a_1 \pm a_2|x)]\tau^{(n)}(x\pm a_0\delta)$$
$$=[(a_0 \pm a_2|x)]\tau^{(n-1)}(x-a_1\delta)\tau^{(n+1)}(x+a_1\delta)-\cdots$$

○ Hypergeometric $\boldsymbol{\tau}$-function (semi-infinite chain)

Definition: A meromorphic ORG τ function $\tau(x)$ on $D_c = \bigsqcup_{n\in\mathbb{Z}} H_{c+n\delta}$ is called a *hypergeometric τ-function* if

$$\tau^{(n)}(x) = 0 \quad (n < 0), \qquad \tau^{(0)}(x) \neq 0.$$

Theorem A (Recursion theorem): *Let $\tau^{(0)}(x)$, $\tau^{(1)}(x)$ be nonzero meromorphic functions on H_c, $H_{c+\delta}$ respectively. Suppose that they satisfy*

$$[(a_0 \pm a_2|x)]\tau^{(0)}(x \pm a_1\delta) = [(a_0 \pm a_1|x)]\tau^{(0)}(x \pm a_2\delta)$$

for any C_3-frame of type II_1, *and*

$$[(a_1 \pm a_2|x)]\tau^{(0)}(x - a_0\delta)\tau^{(1)}(x + a_0\delta) + \cdots = 0$$

for any C_3-frame of type I. *Then these exists a unique a hypergeometric τ-function $\tau(x)$ on D_c such that $\tau^{(0)} = \tau|_{H_c}$ and $\tau^{(1)} = \tau|_{H_{c+\delta}}$.*

Theorem B (Casorati determinant): *Under the assumption of Theorem A, suppose that* $\tau^{(1)}(x)$ *is expressed as* $\tau^{(1)}(x) = \gamma^{(1)}(x)\,\varphi(x)$ *with a nonzero meromorphic function* $\gamma^{(1)}(x)$ *satisfying*

$$[(a_0 + a_2|x)]\gamma^{(1)}(x \pm a_1\delta) = [(a_0 + a_1|x)]\gamma^{(1)}(x \pm a_2\delta)$$

for a C_3*-frame of type* II_1 *with* $(\phi|a_0) = 1$, $(\phi|a_1) = (\phi|a_2) = 0$. *Then the components* $\tau^{(n)}(x)$ *of the hypergeometric* τ*-function* $\tau(x)$ *are expressed as follows in terms of 2-directional Casorati determinants:*

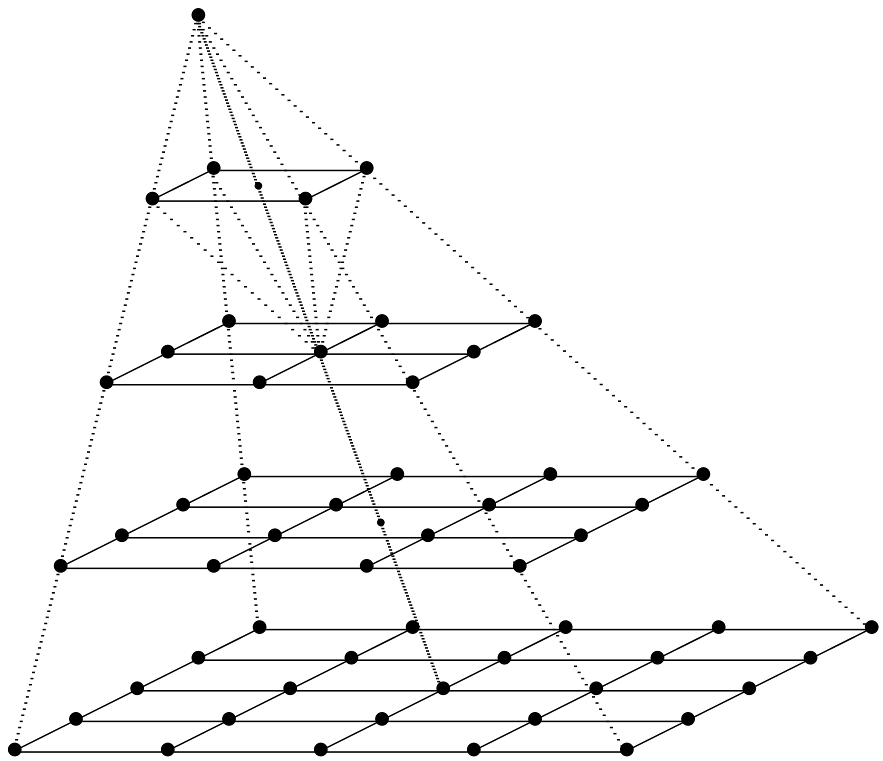
$$\tau^{(n)}(x) = \gamma^{(n)}(x)K^{(n)}(x) \quad (x \in H_{c+n\delta};\ n = 0, 1, 2, \ldots)$$
$$K^{(n)}(x) = \det\left(\varphi_{ij}^{(n)}(x)\right)_{i,j=1}^{n}$$
$$\varphi_{ij}^{(n)}(x) = \varphi(x-(n-1)a_0\delta+(n+1-i-j)a_1\delta+(j-i)a_1\delta) \quad (1 \le i, j \le n).$$

The gauge factors $\gamma^{(n)}(x)$ are determined inductively from $\gamma^{(0)}(x) = \tau^{(0)}(x)$, $\gamma^{(1)}(x)$ by

$$[(a_0 \pm a_2|x)]\gamma^{(n-1)}(x - a_0\delta)\gamma^{(n+1)}(x + a_0\delta) = [(a_1 \pm a_2|x)]\gamma^{(n)}(x \pm a_1\delta).$$

The Toda equation $(\mathrm{II}_1)_n$ corresponds to the *Lewis-Carroll formula* for determinants.

Toda equations produce 2-directional Casorati determinants



$$(\mathrm{II}_1)_n: \ [(a_1 \pm a_2|x)]\tau^{(n-1)}(x - a_0\delta)\tau^{(n+1)}(x + a_0\delta)$$
$$= [(a_0 \pm a_2|x)]\tau^{(n)}(x \pm a_1\delta) - [(a_0 \pm a_1|x)]\tau^{(n)}(x \pm a_2\delta)$$

○ $W(E_7)$-invariant hypergeometric τ-function

We consider the case $[\zeta] = z^{-\frac{1}{2}}\theta(z;p)$, $z = e(\zeta) = e^{2\pi\sqrt{-1}\zeta}$. A typical hypergeometric τ-function for $eP(E_8)$ can be constructed by means of elliptic hypergeometric integrals.

We consider to construct a hypergeometric τ-function on

$$D_\tau = \bigsqcup_{n\in\mathbb{Z}} H_{\tau+n\delta} \quad \text{with} \quad p = e(\tau), \quad q = e(\delta).$$

• $\boldsymbol{\tau^{(0)}(x)}$ The system of first order difference equations for $\tau^{(0)}(x)$ $(x \in H_\tau)$ is solved by a product of triple elliptic gamma functions: for $x \in H_\tau$,

$$\tau^{(0)}(x) = \prod_{0\le i<j\le 7} \Gamma(qu_iu_j;p,q,q), \quad u_i = e(x_i) \ \ (i = 0,1,\ldots,7).$$

in the multiplicative variables, where for $p, q, r \in \mathbb{C}^*$ with $|p|, |q|, |r| < 1$,

$$\Gamma(u;p,q,r) = (u;p,q,r)_\infty(pqr/u;p,q,r)_\infty, \quad (u;p,q,r)_\infty = \prod_{i,j,k=0}^{\infty}(1 - p^iq^jr^ku).$$

• $\boldsymbol{\tau^{(1)}(x)}$ Then, the system of Hirota equations between $\tau^{(0)}(x)$ and $\tau^{(1)}(x)$ is solved by the elliptic hypergeometric integral: for $x \in H_{\tau+\delta}$,

$$\tau^{(1)}(x) = \prod_{0\le i<j\le 7} \Gamma(u_i u_j; p, q, q) \cdot e(-Q(x)) I(u; p, q) \quad (x \in H_{\tau+\delta}),$$

$$Q(x) = \tfrac{1}{2\delta}(x|x), \quad I(u; p, q) = \frac{(p;p)_\infty (q;q)_\infty}{4\pi\sqrt{-1}} \int_C \frac{\prod_{i=0}^{7} \Gamma(u_i z^{\pm 1}; p, q)}{\Gamma(z^{\pm 2}; p, q)} \frac{dz}{z}.$$

Note that the condition $x \in H_{\tau+\delta}$ corresponds to the balancing condition $u_0 u_1 \cdots u_7 = p^2 q^2$ in multiplicative variables. In fact, the system of linear difference equations for $\tau^{(1)}(x)$ reduces to the three term relations

$$[x_j \pm x_k] T^{\delta}_{x_i} J(x) + [x_k \pm x_i] T^{\delta}_{x_j} J(x) + [x_i \pm x_j] T^{\delta}_{x_k} J(x) = 0.$$

for $J(x) = e(-Q(x)) I(u; p, q)$.

• Starting with $\tau^{(0)}(x)$, $\tau^{(1)}(x)$ specified as above, one can construct a $W(E_7)$-invariant hypergeometric function $\tau(x)$.

Theorem C (Determinant formula): *For each* $n = 0, 1, 2, \ldots$, *the* n*th component* $\tau^{(n)}(x)$ *of the* $W(E_7)$*-invariant hypergeometric* τ*-function is expressed as an* $n \times n$ *determinant of elliptic hypergeometric integrals in one variable:*

$$\tau^{(n)}(x) = \gamma^{(n)}(x) \det \left(\varphi_{ij}^{(n)}(x) \right)_{i,j=1}^{n}$$
$$\varphi_{ij}^{(n)}(x) = \varphi(x - (n-1)a_0\delta + (n+1-i-j)a_1\delta + (j-i)a_2\delta)$$

for any C_3*-frame* $\{\pm a_0, \pm a_1, \pm a_2\}$ *of type* II_1 *with* $(\phi|a_0) = 1$, *where*

$$\varphi(x) = e(-\tfrac{1}{2\delta}(x|x)) \cdot \frac{(p;p)_\infty (q;q)_\infty}{4\pi\sqrt{-1}} \int_C \frac{\prod_{i=0}^{7} \Gamma(u_i z^{\pm 1}; p, q)}{\Gamma(z^{\pm 2}; p, q)} \frac{dz}{z}.$$

This 2-directional Casorati determinant can be rewritten into a multiple integral. By Warnaar's elliptic extension of the Krattenthaler determinant, we finally obtain the expression of $\tau^{(n)}(x)$ in terms of the multiple elliptic hypergeometric integral of Rains.

Theorem D (Multiple integral representation): *For each* $n = 0, 1, 2, \ldots$, $\tau^{(n)}(x)$ *is expressed as an elliptic hypergeometric integral of type* BC_n*:*

$$\tau^{(n)}(x) = p^{\binom{n}{2}} \prod_{0 \le i < j \le 7} \Gamma(q^{1-n} u_i u_j; p, q, q) \cdot e(-nQ(x)) \, I^{(n)}(q^{\frac{1}{2}(1-n)} u; p, q, q),$$

$$I^{(n)}(u; p, q, q) = \frac{(p;p)_\infty^n (q;q)_\infty^n}{2^n n! (2\pi\sqrt{-1})^n} \int_{C^n} \prod_{k=1}^{n} \frac{\prod_{i=0}^{7} \Gamma(u_i z_k^{\pm 1}; p, q)}{\Gamma(z_k^{\pm 2}; p, q)} \prod_{1 \le k < l \le n} \theta(z_k^{\pm 1} z_l^{\pm 1}; p) \frac{dz_1 \cdots dz_n}{z_1 \cdots z_n}.$$

References for Part 2

[1] K. Kajiwara, T. Masuda, M. Noumi, Y. Ohta and Y. Yamada: ${}_{10}E_9$ solution to the elliptic Painlevé equation, J. Phys. A. 36(2003), L263–L272.

[2] K. Kajiwara, T. Masuda, M. Noumi, Y. Ohta and Y. Yamada: Point configurations, Cremona transformations and the elliptic difference Painlevé equation, Théories asymptotiques et équations de Painlevé (Angers, juin 2004), Séminaires et Congrès 14(2006), 169–198.

[3] M. Noumi, S. Tsujimoto and Y. Yamada: Padé interpolation for elliptic Painlevé equation, *Symmetries, Integrable Systems and Representations* (K. Iohara, S. Morier-Genoud, B. Rémy Eds.), pp. 463–482, Springer Proceedings in Mathematics and Statistics 40, Springer 2013.

[4] K. Kajiwara, M. Noumi and Y. Yamada: Geometric aspects of Painlevé equations, J. Phys. A: Math. Theor. 50 (2017), 073001 (164pp) (arXiv:1509.08168, 167 pages)

[5] M. Noumi: Remarks on τ-functions for the difference Painlevé equations of type E_8, Advanced Studies in Pure Mathematics **76**(2018), 1–65 (arXiv:16040.6869, 55 pages)